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Noncommutative field theory on homogeneous gravitational waves

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Abstract

We describe an algebraic approach to the time-dependent noncommutative geometry of a six-dimensional Cahen–Wallach pp-wave string background supported by a constant Neveu–Schwarz flux and develop a general formalism to construct and analyse quantum field theories defined thereon. Various star products are derived in closed explicit form and the Hopf algebra of twisted isometries of the plane wave is constructed. Scalar field theories are defined using explicit forms of derivative operators, traces and noncommutative frame fields for the geometry, and various physical features are described. Noncommutative worldvolume field theories of D-branes in the pp-wave background are also constructed.

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1. Introduction and summary

The general construction and analysis of noncommutative gauge theories on curved spacetimes is one of the most important outstanding problems in the applications of noncommutative geometry to string theory. These non-local field theories arise naturally as certain decoupling limits of open string dynamics on D-branes in curved superstring backgrounds in the presence of a non-constant background Neveu–Schwarz B-field. On a generic Poisson manifold M, they are formulated using the Kontsevich star product [47] which is linked to a topological string theory known as the Poisson sigma model [19]. Under suitable conditions, the quantization of D-branes in the Poisson sigma model which wrap coisotropic submanifolds of M, i.e. worldvolumes defined by first-class constraints, may be consistently carried out and related to the deformation quantization in the induced Poisson bracket [20]. Branes defined by second-class constraints may also be treated by quantizing Dirac brackets on the worldvolumes [18].

However, in other concrete string theory settings, most studies of noncommutative gauge theories on curved D-branes have been carried out only within the context of the AdS/CFT correspondence by constructing the branes as solutions in the dual supergravity description of the gauge theory (see for example [5, 15, 16, 39, 41]). It is important to understand how to describe the classical solutions and quantization of these models directly at the field theoretic level in order to better understand to what extent the noncommutative field theories capture the non-local aspects of string theory and quantum gravity, and also to be able to extend the descriptions to more general situations which are not covered by the AdS/CFT correspondence. In this paper, we will investigate worldvolume deformations in the simple example of the Hpp-wave background NW_6 [50], the six-dimensional Cahen–Wallach Lorentzian symmetric space CW_6 [14] supported by a constant null NS–NS background 3-form flux. The spacetime NW_6 lifts to an exact background of ten-dimensional superstring theory by taking the product with an exact four-dimensional background, but we will not write this explicitly. By projecting the transverse space of NW_6 onto a plane, one obtains the four-dimensional Nappi–Witten spacetime NW_4 [52], and occasionally our discussion will pertain to this latter exact string background. Our techniques are presented in a manner which is applicable to a wider class of homogeneous pp-waves supported by a constant Neveu–Schwarz flux.

Open string dynamics on this background is particularly interesting because it has the potential to display a time-dependent noncommutative geometry [32, 39], and hence the noncommutative field theories built on NW_6 can serve as interesting toy models for string cosmology which can be treated for the most part as ordinary field theories. However, this point is rather subtle for the present geometry [32, 40]. A particular gauge choice which leads to a time-dependent noncommutativity parameter breaks conformal invariance of the worldsheet sigma model, i.e. it does not satisfy the Born–Infeld field equations, while a conformally invariant background yields a non-constant but time-independent noncommutativity. In this paper, we will partially clarify this issue. The more complicated noncommutative geometry that we find contains both the transverse space dependent noncommutativity between transverse and light-cone position coordinates of the Hashimoto–Thomas model [40] and the asymptotic time-dependent noncommutativity between transverse space coordinates of the Dolan–Nappi model [32].

The background NW_6 arises as the Penrose–Güven limit [37, 53] of an $\mathrm{AdS}_3 \times S^3$ background [11]. While this limit is a useful tool for understanding various aspects of string dynamics, it is not in general suitable for describing the quantum geometry of embedded D-submanifolds [38]. In the following, we will resort to a more direct quantization of the spacetime NW_6 and its D-submanifolds. We tackle the problem in a purely algebraic way by developing the noncommutative geometry of the universal enveloping algebra of the twisted Heisenberg algebra, whose Lie group $\mathcal{N}$ coincides with the homogeneous spacetime CW_6 in question. While our algebraic approach has the advantage of yielding very explicit constructions of noncommutative field theories in these settings, it also has several limitations. It does not describe the full quantization of the curved spacetime NW_6, but rather only the semi-classical limit of small NS–NS flux θ in which CW_6 approaches flat six-dimensional Minkowski space. This is equivalent to the limit of small light-cone time x^+ for the open string dynamics. In this limit, we can apply the Kontsevich formula to quantize the pertinent Poisson geometry, and hence define noncommutative worldvolume field theories of D-branes. Attempting to quantize the full curved geometry (having $\theta \gg 0$) would bring us deep into the stringy regime [43] wherein a field theoretic analysis would not be possible. The worldvolume deformations in this case are described by nonassociative algebras and variants of quantum group algebras [4, 26], and there is no natural notion of quantization for such geometries.

We will nonetheless emphasize how the effects of curvature manifest themselves in this semi-classical limit.

The spacetime NW_6 is wrapped by non-symmetric D5-branes which can be obtained, as solutions of type II supergravity, from the Penrose–Güven limit of spacetime-filling D5-branes in $AdS_3 \times S^3$ [48]. This paper takes a very detailed look at the first steps towards the construction and analysis of noncommutative worldvolume field theories on these branes. While we deal explicitly only with the case of scalar field theory in detail, leaving the more subtle construction of noncommutative gauge theory for future work, our results provide all the necessary ingredients for analysing generic field theories in these settings. We will also examine the problem of quantizing regularly embedded D-submanifolds in NW_6. The symmetric D-branes wrapping twisted conjugacy classes of the Lie group $\mathcal{N}$ were classified in [61]. Their quantization was analysed in [38] and it was found that, in the semi-classical regime, only the untwisted Euclidean D3-branes support a noncommutative worldvolume geometry. We study these D3-branes as a special case of our more general constructions and find exact agreement with the predictions of the boundary conformal field theory analysis [28]. We also find that the present technique captures the noncommutative worldvolume geometry in a much more natural and tractable way than the foliation of the group $\mathcal{N}$ by quantized coadjoint orbits does [38]. Our analysis is not restricted to symmetric D-branes and can be applied to other D-submanifolds of the spacetime NW_6 as well.

The organization of the remainder of this paper is as follows. In section 2, we describe the twisted Heisenberg algebra, its geometry and the manner in which it may be quantized in the semi-classical limit. In section 3, we construct star products which are equivalent to the Kontsevich product for the pertinent Poisson geometry. These products are much simpler and more tractable than the star product on NW_6 which was constructed in [38] through the noncommutative foliation of NW_6 by D3-branes corresponding to quantized coadjoint orbits. Throughout this paper, we will work with three natural star products which we construct explicitly in closed form. Two of them are canonically related to coordinatizations of the classical pp-wave geometry, while the third one is more natural from the algebraic point of view. We will derive and compare our later results in all three of these star-product deformations.

In section 4, we work out the corresponding generalized Weyl systems [1] for these star products and use them in section 5 to construct the Hopf algebras of twisted isometries [21, 22, 64] of the noncommutative plane wave geometry. In section 6, we use the structure of this Hopf algebra to build derivative operators. In contrast to more conventional approaches [25], these operators are not derivations of the star products but are defined so that they are consistent with the underlying noncommutative algebra of functions. This ensures that the quantum group isometries, which carry the non-trivial curvature content of the spacetime, act consistently on the noncommutative geometry. In section 7, we define integration of fields through a relatively broad class of consistent traces on the noncommutative algebra of functions.

With these general constructions at hand, we proceed in section 8 to analyse as a simple starting example the case of free scalar field theory on the noncommutative spacetime NW_6. The analysis reveals the flat space limiting procedure in a fairly drastic way. To get around this, we introduce noncommutative frame fields which define derivations of the star products [6, 42]. Some potential physical applications in the context of string dynamics in NW_6 [8, 24, 27, 28, 45] are also briefly addressed. Finally, as another application we consider in section 9 the construction of noncommutative worldvolume field theories of D-branes in NW_6 using our general formalism and compare with the quantization of symmetric D-branes which was carried out in [38].

2. Geometry of the twisted Heisenberg algebra

In this section, we will recall the algebraic definition [61] of the six-dimensional gravitational wave NW_6 of Cahen–Wallach type and describe the manner in which its geometry will be quantized in the subsequent sections.

2.1. Definitions

The spacetime NW_6 is defined as the group manifold of the universal central extension of the subgroup $\mathcal{S} := SO(2) \ltimes \mathbb{R}^4$ of the four-dimensional Euclidean group $ISO(4) = SO(4) \ltimes \mathbb{R}^4$. The corresponding simply connected group $\mathcal{N}$ is homeomorphic to six-dimensional Minkowski space $\mathbb{E}^{1,5}$. Its non-semisimple Lie algebra $\mathfrak{n}$ is generated by elements J, T and $\mathsf{P}^i_\pm$, $i = 1, 2$, obeying the non-vanishing commutation relations

$$\left[\mathsf{P}^i_+, \mathsf{P}^j_-\right] = 2\mathrm{i}\delta^{ij}\mathsf{T}, \qquad \left[\mathsf{J}, \mathsf{P}^i_\pm\right] = \pm\mathrm{i}\mathsf{P}^i_\pm. \tag{2.1}$$

This is just the five-dimensional Heisenberg algebra extended by an outer automorphism which rotates the noncommuting coordinates. The twisted Heisenberg algebra may be regarded as defining the harmonic oscillator algebra of a particle moving in two dimensions, with the additional generator J playing the role of the number operator (or equivalently the oscillator Hamiltonian). It is this twisting that will lead to a noncommutative geometry that deviates from the usual Moyal noncommutativity generated by the Heisenberg algebra (see [33, 46, 62] for reviews in the present context). On the other hand, $\mathfrak{n}$ is a solvable algebra whose properties are very tractable. The subgroup $\mathcal{N}_0$ generated by $\mathsf{P}^1_\pm$, J, T is called the Nappi–Witten group and its four-dimensional group manifold is the Nappi–Witten spacetime NW_4 [52].

The most general invariant, non-degenerate symmetric bilinear form $\langle -, -\rangle : \mathfrak{n} \times \mathfrak{n} \to \mathbb{R}$ is defined by the non-vanishing values [52]

$$\left\langle \mathsf{P}^i_+, \mathsf{P}^j_-\right\rangle = 2\delta^{ij}, \qquad \langle \mathsf{J}, \mathsf{T}\rangle = 1, \qquad \langle \mathsf{J}, \mathsf{J}\rangle = b. \tag{2.2}$$

The arbitrary parameter $b \in \mathbb{R}$ can be set to zero by a Lie algebra automorphism of $\mathfrak{n}$. This inner product has Minkowski signature, so that the group manifold of $\mathcal{N}$ possesses a homogeneous, bi-invariant Lorentzian metric defined by the pairing of the Cartan–Maurer left-invariant $\mathfrak{n}$-valued 1-forms $g^{-1}\mathrm{d}g$ for $g \in \mathcal{N}$ as

$$\mathrm{d}s^2 = \langle g^{-1}\,\mathrm{d}g, g^{-1}\,\mathrm{d}g\rangle. \tag{2.3}$$

A generic group element $g \in \mathcal{N}$ may be parametrized as

$$g(u, v, \boldsymbol{a}, \overline{\boldsymbol{a}}) = \mathrm{e}^{a_i \mathsf{P}^i_+ + \overline{a}_i \mathsf{P}^i_-}\,\mathrm{e}^{\theta u \mathsf{J}}\,\mathrm{e}^{\theta^{-1} v \mathsf{T}} \tag{2.4}$$

with $u, v, \theta \in \mathbb{R}$ and $\boldsymbol{a} = (a_1, a_2) \in \mathbb{C}^2$. In these global coordinates, the metric (2.3) reads

$$\mathrm{d}s^2 = 2\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}\boldsymbol{a}\cdot\mathrm{d}\overline{\boldsymbol{a}} + 2\mathrm{i}\theta(\boldsymbol{a}\cdot\mathrm{d}\overline{\boldsymbol{a}} - \overline{\boldsymbol{a}}\cdot\mathrm{d}\boldsymbol{a})\,\mathrm{d}u. \tag{2.5}$$

The metric (2.5) assumes the standard form of the plane wave metric of a conformally flat, indecomposable Cahen–Wallach Lorentzian symmetric spacetime CW_6 in six dimensions [14] upon introduction of Brinkman coordinates [13] $(x^+, x^-, \boldsymbol{z})$ defined by rotating the transverse space at a Larmor frequency as $u = x^+$, $v = x^-$ and $\boldsymbol{a} = \mathrm{e}^{\mathrm{i}\theta x^+/2}\boldsymbol{z}$. In these coordinates, the metric assumes the stationary form

$$\mathrm{d}s^2 = 2\,\mathrm{d}x^+\mathrm{d}x^- + \mathrm{d}\boldsymbol{z}\cdot\mathrm{d}\overline{\boldsymbol{z}} - \tfrac{1}{4}\theta^2|\boldsymbol{z}|^2(\mathrm{d}x^+)^2, \tag{2.6}$$

revealing the pp-wave nature of the geometry. Note that on the null hyperplanes of constant $u = x^+$, the geometry becomes that of flat four-dimensional Euclidean space $\mathbb{E}^4$. This is the geometry appropriate to the Heisenberg subgroup of $\mathcal{N}$ and is what is expected in the Moyal limit when the effects of the extra generator J are turned off.

The spacetime NW_6 is further supported by a Neveu–Schwarz 2-form field B of constant field strength

$$\begin{aligned} H &= -\frac{1}{3}\langle g^{-1}\mathrm{d}g, \mathrm{d}(g^{-1}\,\mathrm{d}g)\rangle = 2\mathrm{i}\theta\,\mathrm{d}x^+\wedge \mathrm{d}\boldsymbol{z}^\top\wedge \mathrm{d}\overline{\boldsymbol{z}} = \mathrm{d}B, \\ B &= -\frac{1}{2}\left\langle g^{-1}\,\mathrm{d}g, \frac{1\!\!1 + \mathrm{Ad}_g}{1\!\!1 - \mathrm{Ad}_g} g^{-1}\mathrm{d}g\right\rangle = 2\mathrm{i}\theta x^+\mathrm{d}\boldsymbol{z}^\top\wedge \mathrm{d}\overline{\boldsymbol{z}}, \end{aligned} \tag{2.7}$$

defined to be non-vanishing only on vectors tangent to the conjugacy class containing $g\in\mathcal{N}$ [3]. It is the presence of this B-field that induces time-dependent noncommutativity of the string background in the presence of D-branes. Because its flux is constant, the noncommutative dynamics in certain kinematical regimes on this space can still be formulated exactly, just like on other symmetric curved noncommutative spaces (see [57] for a review of these constructions in the case of compact group manifolds).

2.2. *Quantization*

We will now begin working our way towards describing how the worldvolumes of D-branes in the spacetime NW_6 are deformed by the non-trivial B-field background. The Seiberg–Witten bi-vector [58] induced by the Neveu–Schwarz background (2.7) and the pp-wave metric G given by (2.6) is

$$\Theta = -(G+B)^{-1}B(G-B). \tag{2.8}$$

Let us introduce the 1-form

$$\Lambda := -\mathrm{i}(\theta^{-1}x_0^- + \theta x^+)(\boldsymbol{z}\cdot \mathrm{d}\overline{\boldsymbol{z}} - \overline{\boldsymbol{z}}\cdot \mathrm{d}\boldsymbol{z}) \tag{2.9}$$

on the null hypersurfaces of constant $x^- = x_0^-$ and compute the corresponding 2-form gauge transformation of the B-field in (2.7) to get

$$B \longmapsto B + \mathrm{d}\Lambda = -\mathrm{i}\theta\,\mathrm{d}x^+\wedge(\boldsymbol{z}\cdot \mathrm{d}\overline{\boldsymbol{z}} - \overline{\boldsymbol{z}}\cdot \mathrm{d}\boldsymbol{z}) + 2\mathrm{i}\theta x_0^-\,\mathrm{d}\overline{\boldsymbol{z}}^\top\wedge \mathrm{d}\boldsymbol{z}. \tag{2.10}$$

The Seiberg–Witten bi-vector in this gauge is given by [38]

$$\Theta = -\frac{2\mathrm{i}\theta}{\theta^2 + (x_0^-)^2}[\theta^2\partial_-\wedge(\boldsymbol{z}\cdot\boldsymbol{\partial} - \overline{\boldsymbol{z}}\cdot\overline{\boldsymbol{\partial}}) + 4x_0^-\boldsymbol{\partial}^\top\wedge\overline{\boldsymbol{\partial}}], \tag{2.11}$$

where $\partial_\pm := \frac{\partial}{\partial x^\pm}$ and $\boldsymbol{\partial} = (\partial^1,\partial^2) := \left(\frac{\partial}{\partial z_1},\frac{\partial}{\partial z_2}\right)$. Since (2.11) is degenerate on the whole NW_6 spacetime, it does not define a symplectic structure. However, one easily checks that it does define a Poisson structure, i.e. Θ is a Poisson bi-vector [38]. In this gauge, one can show that a consistent solution to the Born–Infeld equations of motion on a non-symmetric spacetime-filling D5-brane wrapping NW_6 has vanishing $U(1)$ gauge field flux $F=0$ [40].

In particular, at the special value $x_0^- = \theta$ and with the rescaling $\boldsymbol{z}\to\sqrt{2/\theta\tau}\,\boldsymbol{z}$, the corresponding open string metric [58] $G_{\mathrm{open}} = G - BG^{-1}B$ becomes that of CW_6 in global coordinates (2.5) [38], while the non-vanishing Poisson brackets corresponding to (2.11) read

$$\{z_i,\overline{z}_j\} = 2\mathrm{i}\theta\tau\delta_{ij}, \qquad \{x^-, z_i\} = -\mathrm{i}\theta z_i, \qquad \{x^-,\overline{z}_i\} = \mathrm{i}\theta\overline{z}_i \tag{2.12}$$

for $i,j=1,2$. The Poisson algebra thereby coincides with the Lie algebra $\mathfrak{n}$ in this case and the metric on the branes with the standard curved geometry of the pp-wave. In the semi-classical flat space limit $\theta\to 0$, the quantization of the brackets (2.12) thereby yields a noncommutative worldvolume geometry on D5-branes wrapping NW_6 which can be associated with a quantization of $\mathfrak{n}$ (or more precisely of its dual $\mathfrak{n}^\vee$). In this limit, the corresponding quantization of NW_6 is thus given by the associative Kontsevich star product [47]. Henceforth, with a slight abuse of notation, we will denote the central coordinate τ as the plane wave time coordinate x^+. Our semi-classical quantization will then be valid in the small time limit $x^+ \to 0$.

Our starting point in describing the noncommutative geometry of NW_6 will therefore be at the algebraic level. We will consider the deformation quantization of the dual $\mathfrak{n}^\vee$ to the Lie algebra $\mathfrak{n}$. Naively, one may think that the easiest way to carry this out is to compute star products on the pp-wave by taking the Penrose limits of the standard ones on S^3 and AdS_3 (or equivalently by contracting the standard quantizations of the Lie algebras $su(2)$ and $sl(2,\mathbb{R})$). However, some quick calculations show that the induced star products obtained in this way are divergent in the infinite volume limit, and the reason why is simple. While the standard Inönü–Wigner contractions hold at the level of the Lie algebras [61], they need not necessarily map the corresponding universal enveloping algebras, on which the quantizations are performed. This is connected to the phenomenon that twisted conjugacy classes of branes are not necessarily related by the Penrose–Güven limit [38]. We must therefore resort to a more direct approach to quantizing the spacetime NW_6.

For notational ease, we will write the algebra $\mathfrak{n}$ in the generic form

$$[\mathsf{X}_a, \mathsf{X}_b] = \mathrm{i}\theta C_{ab}^c \mathsf{X}_c, \tag{2.13}$$

where $(\mathsf{X}_a) := \theta\big(\mathsf{J}, \mathsf{T}, \mathsf{P}_\pm^i\big)$ are the generators of $\mathfrak{n}$ and the structure constants C_{ab}^c can be gleamed off from (2.1). The algebra (2.13) can be regarded as a formal deformation quantization of the Kirillov–Kostant Poisson bracket on $\mathfrak{n}^\vee$ in the standard coadjoint orbit method. Let us identify $\mathfrak{n}^\vee$ as the vector space $\mathbb{R}^6$ with basis $\mathsf{X}_a^\vee := \langle \mathsf{X}_a, - \rangle : \mathfrak{n} \to \mathbb{R}$ dual to the X_a. In the algebra of polynomial functions $\mathbb{C}(\mathfrak{n}^\vee) = \mathbb{C}(\mathbb{R}^6)$, we may then identify the generators X_a themselves with the coordinate functions

$$\begin{aligned} &\mathsf{X}_\mathsf{J}(\boldsymbol{x}) = x_\mathsf{T} = x^-, && \mathsf{X}_\mathsf{T}(\boldsymbol{x}) = x_\mathsf{J} = x^+, \\ &\mathsf{X}_{\mathsf{P}_+^i}(\boldsymbol{x}) = 2x_{\mathsf{P}_-^i} = 2\bar{z}_i, && \mathsf{X}_{\mathsf{P}_-^i}(\boldsymbol{x}) = 2x_{\mathsf{P}_+^i} = 2z_i \end{aligned} \tag{2.14}$$

for any $\boldsymbol{x} \in \mathfrak{n}^\vee$ with component x_a in the $\mathsf{X}_a^\vee$ direction. These functions generate the whole coordinate algebra and their Poisson bracket Θ is defined by

$$\Theta(\mathsf{X}_a, \mathsf{X}_b)(\boldsymbol{x}) = \boldsymbol{x}([\mathsf{X}_a, \mathsf{X}_b]), \quad \forall\, \boldsymbol{x} \in \mathfrak{n}^\vee. \tag{2.15}$$

Therefore, when viewed as functions on $\mathbb{R}^6$ the Lie algebra generators have a Poisson bracket given by the Lie bracket, and their quantization is provided by (2.13) with deformation parameter θ. In the next section, we will explore various aspects of this quantization and derive several (equivalent) star products on $\mathfrak{n}^\vee$.

3. Gutt products

The formal completion of the space of polynomials $\mathbb{C}(\mathfrak{n}^\vee)$ is the algebra $C^\infty(\mathfrak{n}^\vee)$ of smooth functions on $\mathfrak{n}^\vee$. There is a natural way to construct a star product on the cotangent bundle $T^*\mathcal{N} \cong \mathcal{N} \times \mathfrak{n}^\vee$, which naturally induces an associative product on $C^\infty(\mathfrak{n}^\vee)$. This induced product is called the Gutt product [36]. The Poisson bracket defined by (2.15) naturally extends to a Poisson structure $\Theta : C^\infty(\mathfrak{n}^\vee) \times C^\infty(\mathfrak{n}^\vee) \to C^\infty(\mathfrak{n}^\vee)$ defined by the Kirillov–Kostant bi-vector

$$\Theta = \tfrac{1}{2} C_{ab}^c x_c \partial^a \wedge \partial^b, \tag{3.1}$$

where $\partial^a := \frac{\partial}{\partial x_a}$. This coincides with the Seiberg–Witten bi-vector in the limits described in section 2.2. The Gutt product constructs a quantization of this Poisson structure. It is equivalent to the Kontsevich star product in this case [31], and by construction it keeps that part of the Kontsevich formula which is associative [60]. In general, within the present context, the Gutt and Kontsevich deformation quantizations are only identical for nilpotent Lie algebras [44].

The algebra $\mathbb{C}(\mathfrak{n}^\vee)$ of polynomial functions on the dual to the Lie algebra is naturally isomorphic to the symmetric tensor algebra $S(\mathfrak{n})$ of $\mathfrak{n}$. By the Poincaré–Birkhoff–Witt theorem, there is a natural isomorphism $\Omega : S(\mathfrak{n}) \to U(\mathfrak{n})$ with the universal enveloping algebra $U(\mathfrak{n})$ of $\mathfrak{n}$. Using the above identifications, this extends to a canonical isomorphism

$$\Omega : C^\infty(\mathbb{R}^6) \longrightarrow \overline{U(\mathfrak{n})^{\mathbb{C}}} \tag{3.2}$$

defined by specifying an ordering for the elements of the basis of monomials for $S(\mathfrak{n})$, where $\overline{U(\mathfrak{n})^{\mathbb{C}}}$ denotes a formal completion of the complexified universal enveloping algebra $U(\mathfrak{n})^{\mathbb{C}} := U(\mathfrak{n}) \otimes \mathbb{C}$. Denoting this ordering by $\substack{\circ \\ \circ} - \substack{\circ \\ \circ}$, we may write this isomorphism symbolically as

$$\Omega\big(x_{a_1} \cdots x_{a_n}\big) = \substack{\circ \\ \circ}\mathsf{X}_{a_1} \cdots \mathsf{X}_{a_n}\substack{\circ \\ \circ}. \tag{3.3}$$

The original Gutt construction [36] takes the isomorphism Ω on $S(\mathfrak{n})$ to be symmetrization of monomials. In this case, $\Omega(f)$ is usually called the Weyl symbol of $f \in C^\infty(\mathbb{R}^6)$ and the symmetric ordering $\substack{\circ \\ \circ} - \substack{\circ \\ \circ}$ of symbols $\Omega(f)$ is called Weyl ordering. In the following, we shall work with three natural orderings appropriate to the algebra $\mathfrak{n}$.

The isomorphism (3.2) can be used to transport the algebraic structure on the universal enveloping algebra $U(\mathfrak{n})$ of $\mathfrak{n}$ to the algebra of smooth functions on $\mathfrak{n}^\vee \cong \mathbb{R}^6$ and give the star product defined by

$$f \star g := \Omega^{-1}\big(\substack{\circ \\ \circ}\Omega(f) \cdot \Omega(g)\substack{\circ \\ \circ}\big), \qquad f, g \in C^\infty(\mathbb{R}^6). \tag{3.4}$$

The product on the right-hand side of formula (3.4) is taken in $U(\mathfrak{n})$, and it follows that $\star$ defines an associative, noncommutative product. Moreover, it represents a deformation quantization of the Kirillov–Kostant Poisson structure on $\mathfrak{n}^\vee$, in the sense that

$$[x, y]_\star := x \star y - y \star x = \mathrm{i}\theta\Theta(x, y), \qquad x, y \in \mathbb{C}_{(1)}(\mathfrak{n}^\vee), \tag{3.5}$$

where $\mathbb{C}_{(1)}(\mathfrak{n}^\vee)$ is the subspace of homogeneous polynomials of degree 1 on $\mathfrak{n}^\vee$. In particular, the Lie algebra relations (2.13) are reproduced by star commutators of the coordinate functions as

$$[x_a, x_b]_\star = \mathrm{i}\theta C_{ab}^{c} x_c, \tag{3.6}$$

in accordance with the Poisson brackets (2.12) and definition (2.15).

Let us now describe how to write the star product (3.4) explicitly in terms of a bi-differential operator $\hat{\mathcal{D}} : C^\infty(\mathfrak{n}^\vee) \times C^\infty(\mathfrak{n}^\vee) \to C^\infty(\mathfrak{n}^\vee)$ [44]. Using the Kirillov–Kostant Poisson structure as before, we identify the generators of $\mathfrak{n}$ as coordinates on $\mathfrak{n}^\vee$. This establishes, for small $s \in \mathbb{R}$, a one-to-one correspondence between group elements $\mathrm{e}^{s\mathsf{X}}$, $\mathsf{X} \in \mathfrak{n}$ and functions e^{sx} on $\mathfrak{n}^\vee$. Pulling back the group multiplication of elements $\mathrm{e}^{s\mathsf{X}} \in \mathcal{N}$ via this correspondence induces a bi-differential operator $\hat{\mathcal{D}}$ acting on the functions e^{sx}. Since these functions separate the points on $\mathfrak{n}^\vee$, this extends to an operator on the whole of $C^\infty(\mathfrak{n}^\vee)$.

To apply this construction explicitly, we use the following trick [6, 49] which will also prove useful for later considerations. By restricting to an appropriate Schwartz subspace of functions $f \in C^\infty(\mathbb{R}^6)$, we may use a Fourier representation

$$f(\boldsymbol{x}) = \int_{\mathbb{R}^6} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^6} \tilde{f}(\boldsymbol{k})\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{3.7}$$

This establishes a correspondence between (Schwartz) functions on $\mathfrak{n}^\vee$ and elements of the complexified group $\mathcal{N}^{\mathbb{C}} := \mathcal{N} \otimes \mathbb{C}$. The products of symbols $\Omega(f)$ may be computed using (3.3), and the star product (3.4) can be represented in terms of a product of group elements in $\mathcal{N}^{\mathbb{C}}$ as

$$f \star g = \int_{\mathbb{R}^6} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^6} \int_{\mathbb{R}^6} \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^6} \tilde{f}(\boldsymbol{k})\tilde{g}(\boldsymbol{q})\Omega^{-1}\left({}^{\circ}_{\circ}{}^{\circ}_{\circ}\, \mathrm{e}^{\mathrm{i}k^a \mathsf{X}_a} \, {}^{\circ}_{\circ} \cdot {}^{\circ}_{\circ}\, \mathrm{e}^{\mathrm{i}q^a \mathsf{X}_a} \, {}^{\circ}_{\circ}{}^{\circ}_{\circ} \right). \tag{3.8}$$

Using the Baker–Campbell–Hausdorff formula, to be discussed below, we may write

$${}^{\circ}_{\circ}{}^{\circ}_{\circ}\, \mathrm{e}^{\mathrm{i}k^a \mathsf{X}_a} \, {}^{\circ}_{\circ} \cdot {}^{\circ}_{\circ}\, \mathrm{e}^{\mathrm{i}q^a \mathsf{X}_a} \, {}^{\circ}_{\circ}{}^{\circ}_{\circ} = {}^{\circ}_{\circ}\, \mathrm{e}^{\mathrm{i}D^a(\boldsymbol{k},\boldsymbol{q})\mathsf{X}_a} \, {}^{\circ}_{\circ} \tag{3.9}$$

for some function $\boldsymbol{D} = (D^a) : \mathbb{R}^6 \times \mathbb{R}^6 \to \mathbb{R}^6$. This enables us to rewrite the star product (3.8) in terms of a bi-differential operator $f \star g := \hat{\mathcal{D}}(f, g)$ given explicitly by

$$f \star g = f \mathrm{e}^{\mathrm{i}\boldsymbol{x}\cdot[\boldsymbol{D}(-\mathrm{i}\overleftarrow{\boldsymbol{\partial}}, -\mathrm{i}\overrightarrow{\boldsymbol{\partial}}) + \mathrm{i}\overleftarrow{\boldsymbol{\partial}} + \mathrm{i}\overrightarrow{\boldsymbol{\partial}}]} g \tag{3.10}$$

with $\boldsymbol{\partial} := (\partial^a)$. In particular, the star products of the coordinate functions themselves may be computed from the formula

$$x_a \star x_b = -\frac{\partial}{\partial k^a}\frac{\partial}{\partial q^b} \mathrm{e}^{\mathrm{i}\boldsymbol{D}(\boldsymbol{k},\boldsymbol{q})\cdot\boldsymbol{x}} \Big|_{\boldsymbol{k}=\boldsymbol{q}=\boldsymbol{0}}. \tag{3.11}$$

Finally, let us describe how to explicitly compute the functions $D^a(\boldsymbol{k}, \boldsymbol{q})$ in (3.9). For this, we consider the Dynkin form of the Baker–Campbell–Hausdorff formula which is given for $\mathsf{X}, \mathsf{Y} \in \mathfrak{n}$ by

$$\mathrm{e}^{\mathsf{X}}\, \mathrm{e}^{\mathsf{Y}} = \mathrm{e}^{H(\mathsf{X}:\mathsf{Y})}, \tag{3.12}$$

where $H(\mathsf{X} : \mathsf{Y}) = \sum_{n\geqslant 1} H_n(\mathsf{X} : \mathsf{Y}) \in \mathfrak{n}$ is generically an infinite series whose terms may be calculated through the recurrence relation

$$\begin{aligned}(n+1)H_{n+1}(\mathsf{X} : \mathsf{Y}) &= \frac{1}{2}[\mathsf{X} - \mathsf{Y}, H_n(\mathsf{X} : \mathsf{Y})] \\ &\quad + \sum_{p=1}^{\lfloor n/2 \rfloor} \frac{B_{2p}}{(2p)!} \sum_{\substack{k_1,\ldots,k_{2p}>0 \\ k_1+\cdots+k_{2p}=n}} \left[H_{k_1}(\mathsf{X} : \mathsf{Y}), \left[\ldots, \left[H_{k_{2p}}(\mathsf{X} : \mathsf{Y}), \mathsf{X} + \mathsf{Y}\right]\ldots\right]\right]\end{aligned} \tag{3.13}$$

with $H_1(\mathsf{X} : \mathsf{Y}) := \mathsf{X} + \mathsf{Y}$. The coefficients B_{2p} are the Bernoulli numbers which are defined by the generating function

$$\frac{s}{1 - \mathrm{e}^{-s}} - \frac{s}{2} - 1 = \sum_{p=1}^{\infty} \frac{B_{2p}}{(2p)!} s^{2p}. \tag{3.14}$$

The first few terms of formula (3.12) may be written explicitly as

$$\begin{aligned} &H_1(\mathsf{X} : \mathsf{Y}) = \mathsf{X} + \mathsf{Y}, && H_2(\mathsf{X} : \mathsf{Y}) = \tfrac{1}{2}[\mathsf{X}, \mathsf{Y}], \\ &H_3(\mathsf{X} : \mathsf{Y}) = \tfrac{1}{12}[\mathsf{X}, [\mathsf{X}, \mathsf{Y}]] - \tfrac{1}{12}[\mathsf{Y}, [\mathsf{X}, \mathsf{Y}]], && H_4(\mathsf{X} : \mathsf{Y}) = -\tfrac{1}{24}[\mathsf{Y}, [\mathsf{X}, [\mathsf{X}, \mathsf{Y}]]]. \end{aligned} \tag{3.15}$$

Terms in the series grow increasingly complicated due to the sum over partitions in (3.13), and in general there is no closed symbolic form, as in the case of the Moyal product based on the ordinary Heisenberg algebra, for the functions $D^a(\boldsymbol{k}, \boldsymbol{q})$ appearing in (3.9). However, at least for certain ordering prescriptions, the solvability of the Lie algebra $\mathfrak{n}$ enables one to find explicit expressions for the star product (3.10) in this fashion. We will now proceed to construct three such products.

3.1. Time ordering

The simplest Gutt product is obtained by choosing a 'time ordering' prescription in (3.3) whereby all factors of the time translation generator J occur to the far right in any monomial in $U(\mathfrak{n})$. It coincides precisely with the global coordinatization (2.4) of the Cahen–Wallach spacetime, and written on elements of the complexified group $\mathcal{N}^{\mathbb{C}}$ it is defined by

$$\Omega_*(\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}) = {}^*_*\, \mathrm{e}^{\mathrm{i}k^a\mathsf{X}_a}\, {}^*_* := \mathrm{e}^{\mathrm{i}(p_i^+\mathsf{P}_+^i + p_i^-\mathsf{P}_-^i)}\, \mathrm{e}^{\mathrm{i}j\mathsf{J}}\, \mathrm{e}^{\mathrm{i}t\mathsf{T}}, \tag{3.16}$$

where we have denoted $\boldsymbol{k} := (j, t, \boldsymbol{p}^\pm)$ with $j, t \in \mathbb{R}$ and $\boldsymbol{p}^\pm = \overline{\boldsymbol{p}^\mp} = \left(p_1^\pm, p_2^\pm\right) \in \mathbb{C}^2$. To calculate the corresponding star product $*$, we have to compute the group products

$$^{*\,*}_{*\,*}\, \mathrm{e}^{\mathrm{i}k^a\mathsf{X}_a}\, {}^*_* \cdot {}^*_*\, \mathrm{e}^{\mathrm{i}k'^a\mathsf{X}_a}\, {}^{*\,*}_{*\,*} = {}^*_*\, \mathrm{e}^{\mathrm{i}(p_i^+\mathsf{P}_+^i + p_i^-\mathsf{P}_-^i)}\, \mathrm{e}^{\mathrm{i}j\mathsf{J}}\, \mathrm{e}^{\mathrm{i}t\mathsf{T}} \times \mathrm{e}^{\mathrm{i}(p_i'^+\mathsf{P}_+^i + p_i'^-\mathsf{P}_-^i)}\, \mathrm{e}^{\mathrm{i}j'\mathsf{J}}\, \mathrm{e}^{\mathrm{i}t'\mathsf{T}}\, {}^*_*. \tag{3.17}$$

The simplest way to compute these products is to realize the six-dimensional Lie algebra $\mathfrak{n}$ as a central extension of the subalgebra $\mathfrak{s} = so(2) \ltimes \mathbb{R}^4$ of the four-dimensional Euclidean algebra $iso(4) = so(4) \ltimes \mathbb{R}^4$ [61, 35]. Regarding $\mathbb{R}^4$ as $\mathbb{C}^2$ (with respect to a chosen complex structure), for generic $\theta \neq 0$ the generators of $\mathfrak{n}$ act on $\boldsymbol{w} \in \mathbb{C}^2$ according to the affine transformations $\mathrm{e}^{\mathrm{i}j\mathsf{J}} \cdot \boldsymbol{w} = \mathrm{e}^{-\theta j}\boldsymbol{w}$ and $\mathrm{e}^{\mathrm{i}(p_i^+\mathsf{P}_+^i + p_i^-\mathsf{P}_-^i)} \cdot \boldsymbol{w} = \boldsymbol{w} + \mathrm{i}\theta\boldsymbol{p}^-$, corresponding to a combined rotation in the (12), (34) planes and translations in $\mathbb{R}^4 \cong \mathbb{C}^2$. The central element generates an abstract one-parameter subgroup acting as $\mathrm{e}^{\mathrm{i}t\mathsf{T}} \cdot \boldsymbol{w} = \mathrm{e}^{-\theta t}\boldsymbol{w}$ in this representation. From this action we can read off the group multiplication laws

$$\mathrm{e}^{\mathrm{i}j\mathsf{J}}\, \mathrm{e}^{\mathrm{i}j'\mathsf{J}} = \mathrm{e}^{\mathrm{i}(j+j')\mathsf{J}}, \tag{3.18}$$

$$\mathrm{e}^{\mathrm{i}j\mathsf{J}}\, \mathrm{e}^{\mathrm{i}(p_i^+\mathsf{P}_+^i + p_i^-\mathsf{P}_-^i)} = \mathrm{e}^{\mathrm{i}(\mathrm{e}^{-\theta j} p_i^+\mathsf{P}_+^i + \mathrm{e}^{\theta j} p_i^-\mathsf{P}_-^i)}\, \mathrm{e}^{\mathrm{i}j\mathsf{J}}, \tag{3.19}$$

$$\mathrm{e}^{\mathrm{i}(p_i^+\mathsf{P}_+^i + p_i^-\mathsf{P}_-^i)}\, \mathrm{e}^{\mathrm{i}(p_i'^+\mathsf{P}_+^i + p_i'^-\mathsf{P}_-^i)} = \mathrm{e}^{\mathrm{i}[(p_i^+ + p_i'^+)\mathsf{P}_+^i + (p_i^- + p_i'^-)\mathsf{P}_-^i]}\, \mathrm{e}^{2\theta\,\mathrm{Im}(\boldsymbol{p}^+\cdot\boldsymbol{p}'^-)\mathsf{T}} \tag{3.20}$$

where formula (3.19) displays the semi-direct product nature of the Euclidean group, while (3.20) displays the group cocycle of the projective representation of the subgroup $\mathcal{S}$ of $ISO(4)$, arising from the central extension, which makes the translation algebra noncommutative and is computed from the Baker–Campbell–Hausdorff formula.

Using (3.18)–(3.20) we may now compute the products (3.17) and one finds

$$^{*\,*}_{*\,*}\, \mathrm{e}^{\mathrm{i}k^a\mathsf{X}_a}\, {}^*_* \cdot {}^*_*\, \mathrm{e}^{\mathrm{i}k'^a\mathsf{X}_a}\, {}^{*\,*}_{*\,*} = \mathrm{e}^{\mathrm{i}[(p_i^+ + \mathrm{e}^{-\theta j} p_i'^+)\mathsf{P}_+^i + (p_i^- + \mathrm{e}^{\theta j} p_i'^-)\mathsf{P}_-^i]}\, \mathrm{e}^{\mathrm{i}(j+j')\mathsf{J}} \times \mathrm{e}^{\mathrm{i}[t+t'-\theta(\mathrm{e}^{\theta j}\boldsymbol{p}^+\cdot\boldsymbol{p}'^- - \mathrm{e}^{-\theta j}\boldsymbol{p}^-\cdot\boldsymbol{p}'^+)]\mathsf{T}}. \tag{3.21}$$

From (3.11) we may compute the star products between the coordinate functions on $\mathfrak{n}^\vee$ and easily verify the commutation relations of the algebra $\mathfrak{n}$,

$$\begin{aligned}
&x_a * x_a = (x_a)^2, && x_a * x^+ = x^+ * x_a = x_a x^+,\\
&z_1 * z_2 = z_2 * z_1 = z_1 z_2, && \overline{z}_1 * \overline{z}_2 = \overline{z}_2 * \overline{z}_1 = \overline{z}_1\overline{z}_2,\\
&x^- * z_i = x^- z_i - \mathrm{i}\theta z_i, && z_i * x^- = x^- z_i,\\
&x^- * \overline{z}_i = x^- \overline{z}_i + \mathrm{i}\theta\overline{z}_i, && \overline{z}_i * x^- = x^- \overline{z}_i,\\
&z_i * \overline{z}_i = z_i\overline{z}_i - \mathrm{i}\theta x^+, && \overline{z}_i * z_i = z_i\overline{z}_i + \mathrm{i}\theta x^+,
\end{aligned} \tag{3.22}$$

with $a = 1, \ldots, 6$ and $i = 1, 2$. From (3.9) and (3.10), we find the star product $*$ of generic functions $f, g \in C^\infty(\mathfrak{n}^\vee)$ given by

$$\begin{aligned}
f * g = \mu \circ \exp[&\mathrm{i}\theta x^+(\mathrm{e}^{-\mathrm{i}\theta\partial_-}\boldsymbol{\partial}^\top \otimes \overline{\boldsymbol{\partial}} - \mathrm{e}^{\mathrm{i}\theta\partial_-}\overline{\boldsymbol{\partial}}^\top \otimes \boldsymbol{\partial})\\
&+ \overline{z}_i(\mathrm{e}^{\mathrm{i}\theta\partial_-} - 1) \otimes \partial^i + z_i(\mathrm{e}^{-\mathrm{i}\theta\partial_-} - 1) \otimes \overline{\partial}^i]f \otimes g,
\end{aligned} \tag{3.23}$$

where $\mu(f\otimes g) = fg$ is the pointwise product. To second order in the deformation parameter θ, we obtain

$$
\begin{aligned}
f * g = fg &- \mathrm{i}\theta[x^+(\overline{\boldsymbol{\partial}} f\cdot\boldsymbol{\partial} g - \boldsymbol{\partial} f\cdot\overline{\boldsymbol{\partial}} g) - \overline{\boldsymbol{z}}\cdot\partial_- f\boldsymbol{\partial} g + \boldsymbol{z}\cdot\partial_- f\overline{\boldsymbol{\partial}} g] \\
&- \theta^2\sum_{i=1,2}\bigg[\frac{1}{2}(x^+)^2((\partial^i)^2 f(\overline{\partial}^i)^2 g - 2\overline{\partial}^i\partial^i f\overline{\partial}^i\partial^i g + (\overline{\partial}^i)^2 f(\partial^i)^2 g) \\
&- x^+(\partial^i\partial_- f\overline{\partial}^i g - \overline{\partial}^i\partial_- f\partial^i g) - x^+\overline{z}_i(\overline{\partial}^i\partial_- f(\partial^i)^2 g - \partial^i\partial_- f\overline{\partial}^i\partial^i g) \\
&+ x^+ z_i(\overline{\partial}^i\partial_- f\overline{\partial}^i\partial^i g - \partial^i\partial_- f(\overline{\partial}^i)^2 g) - \overline{z}_i z_i\partial_-^2 f\overline{\partial}^i\partial^i g \\
&+ \frac{1}{2}\left(\overline{z}_i^2\partial_-^2 f(\partial^i)^2 g + \overline{z}_i\partial_-^2 f\partial^i g + z_i\partial_-^2 f\overline{\partial}^i g + z_i^2\partial_-^2 f(\overline{\partial}^i)^2 g\right)\bigg] + O(\theta^3).
\end{aligned}
\tag{3.24}
$$

3.2. Symmetric time ordering

Our next Gutt product is obtained by taking a 'symmetric time ordering' whereby any monomial in $U(\mathfrak{n})$ is the symmetric sum over the two time orderings obtained by placing J to the far right and to the far left. This ordering is induced by the group contraction of $U(1)\times SU(2)$ onto the Nappi–Witten group $\mathcal{N}_0$ [27], and it thereby induces the coordinatization of NW_4 that is obtained from the Penrose–Güven limit of the spacetime $S^{1,0}\times S^3$, i.e. it coincides with the Brinkman coordinatization of the Cahen–Wallach spacetime. On elements of $\mathcal{N}^{\mathbb{C}}$ it is defined by

$$
\Omega_\bullet(\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}) = {}^\bullet_\bullet\mathrm{e}^{\mathrm{i}k^a\mathsf{X}_a}{}^\bullet_\bullet := \mathrm{e}^{\frac{\mathrm{i}}{2}j\mathsf{J}}\,\mathrm{e}^{\mathrm{i}(p_i^+\mathsf{P}_+^i + p_i^-\mathsf{P}_-^i)}\,\mathrm{e}^{\frac{\mathrm{i}}{2}j\mathsf{J}}\,\mathrm{e}^{\mathrm{i}t\mathsf{T}}. \tag{3.25}
$$

From (3.18)–(3.20) we can again easily compute the required group products to get

$$
\begin{aligned}
{}^\bullet_\bullet{}^\bullet_\bullet\mathrm{e}^{\mathrm{i}k^a\mathsf{X}_a}{}^\bullet_\bullet\cdot{}^\bullet_\bullet\mathrm{e}^{\mathrm{i}k'^a\mathsf{X}_a}{}^\bullet_\bullet{}^\bullet_\bullet &= \exp\left(\frac{\mathrm{i}}{2}(j+j')\mathsf{J}\right) \\
&\quad\times\exp\left(\mathrm{i}\left[\left(\mathrm{e}^{\frac{\theta}{2}j'}p_i^+ + \mathrm{e}^{-\frac{\theta}{2}j}p_i'^+\right)\mathsf{P}_+^i + \left(\mathrm{e}^{-\frac{\theta}{2}j'}p_i^- + \mathrm{e}^{\frac{\theta}{2}j}p_i'^-\right)\mathsf{P}_-^i\right]\right) \\
&\quad\times\exp\left(\frac{\mathrm{i}}{2}(j+j')\mathsf{J}\right)\exp\left(\mathrm{i}\left[t+t'-\theta\left(\mathrm{e}^{\frac{\theta}{2}(j+j')}\boldsymbol{p}^+\cdot\boldsymbol{p}'^- - \mathrm{e}^{-\frac{\theta}{2}(j+j')}\boldsymbol{p}^-\cdot\boldsymbol{p}'^+\right)\right]\mathsf{T}\right).
\end{aligned}
\tag{3.26}
$$

With the same conventions as above, from (3.11) we may now compute the star products $\bullet$ between the coordinate functions on $\mathfrak{n}^\vee$ and again verify the commutation relations of the algebra $\mathfrak{n}$,

$$
\begin{aligned}
&x_a\bullet x_a = (x_a)^2, && x_a\bullet x^+ = x^+\bullet x_a = x_a x^+, \\
&z_1\bullet z_2 = z_2\bullet z_1 = z_1 z_2, && \overline{z}_1\bullet\overline{z}_2 = \overline{z}_2\bullet\overline{z}_1 = \overline{z}_1\overline{z}_2, \\
&x^-\bullet z_i = x^- z_i - \frac{\mathrm{i}}{2}\theta z_i, && z_i\bullet x^- = x^- z_i + \frac{\mathrm{i}}{2}\theta z_i, \\
&x^-\bullet\overline{z}_i = x^-\overline{z}_i + \frac{\mathrm{i}}{2}\theta\overline{z}_i, && \overline{z}_i\bullet x^- = x^-\overline{z}_i - \frac{\mathrm{i}}{2}\theta\overline{z}_i, \\
&z_i\bullet\overline{z}_i = z_i\overline{z}_i - \mathrm{i}\theta x^+, && \overline{z}_i\bullet z_i = z_i\overline{z}_i + \mathrm{i}\theta x^+.
\end{aligned}
\tag{3.27}
$$

From (3.9) and (3.10) we find for generic functions the formula

$$
\begin{aligned}
f\bullet g = \mu\circ\exp\Big\{&\mathrm{i}\theta x^+\left(\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}\partial_-}\boldsymbol{\partial}^\top\otimes\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}\partial_-}\overline{\boldsymbol{\partial}} - \mathrm{e}^{\frac{\mathrm{i}\theta}{2}\partial_-}\overline{\boldsymbol{\partial}}^\top\otimes\mathrm{e}^{\frac{\mathrm{i}\theta}{2}\partial_-}\boldsymbol{\partial}\right) \\
&+\overline{z}_i\left[\partial^i\otimes\left(\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}\partial_-}-1\right) + \left(\mathrm{e}^{\frac{\mathrm{i}\theta}{2}\partial_-}-1\right)\otimes\partial^i\right] \\
&+z_i\left[\overline{\partial}^i\otimes\left(\mathrm{e}^{\frac{\mathrm{i}\theta}{2}\partial_-}-1\right) + \left(\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}\partial_-}-1\right)\otimes\overline{\partial}^i\right]\Big\}f\otimes g.
\end{aligned}
\tag{3.28}
$$

To second order in θ, we obtain

$$
\begin{aligned}
f \bullet g = fg - \frac{\mathrm{i}}{2}\theta[2x^+(\overline{\boldsymbol{\partial}} f \cdot \boldsymbol{\partial} g - \boldsymbol{\partial} f \cdot \overline{\boldsymbol{\partial}} g) \\
&+ \overline{\boldsymbol{z}} \cdot (\boldsymbol{\partial} f \partial_- g - \partial_- f \boldsymbol{\partial} g) + \boldsymbol{z} \cdot (\partial_- f \overline{\boldsymbol{\partial}} g - \overline{\boldsymbol{\partial}} f \partial_- g)] \\
&- \frac{1}{2}\theta^2 \sum_{i=1,2} \bigg[(x^+)^2((\overline{\partial}^i)^2 f (\partial^i)^2 g + (\partial^i)^2 f (\overline{\partial}^i)^2 g - 2\overline{\partial}^i \partial^i f \overline{\partial}^i \partial^i g) \\
&- x^+(\partial^i f \overline{\partial}^i \partial_- g + \overline{\partial}^i f \partial^i \partial_- g + \overline{\partial}^i \partial_- f \partial^i g + \partial^i \partial_- f \overline{\partial}^i g) \\
&+ x^+ \overline{z}_i (\overline{\partial}^i \partial^i f \partial^i \partial_- g - \overline{\partial}^i \partial_- f (\partial^i)^2 g + \partial^i \partial_- f \overline{\partial}^i \partial^i g - (\partial^i)^2 f \overline{\partial}^i \partial_- g) \\
&+ x^+ z_i (\overline{\partial}^i \partial^i f \overline{\partial}^i \partial_- g - \partial^i \partial_- f (\overline{\partial}^i)^2 g + \overline{\partial}^i \partial_- f \overline{\partial}^i \partial^i g - (\overline{\partial}^i)^2 f \partial^i \partial_- g) \\
&+ \frac{1}{2}\overline{z}_i z_i \big(\overline{\partial}^i \partial_- f \partial^i \partial_- g + \partial^i \partial_- f \overline{\partial}^i \partial_- g - \partial_-^2 f \overline{\partial}^i \partial^i g - \overline{\partial}^i \partial^i f \partial_-^2 g \big) \\
&+ \frac{1}{4}\overline{z}_i^2 \big((\partial^i)^2 f \partial_-^2 g - 2\partial^i \partial_- f \partial^i \partial_- g + \partial_-^2 f (\partial^i)^2 g \big) \\
&+ \frac{1}{4} z_i^2 \big((\overline{\partial}^i)^2 f \partial_-^2 g - 2\overline{\partial}^i \partial_- f \overline{\partial}^i \partial_- g + \partial_-^2 f (\overline{\partial}^i)^2 g \big) \\
&+ \frac{1}{4}\overline{z}_i \big(\partial^i f \partial_-^2 g + \partial_-^2 f \partial^i g \big) + \frac{1}{4} z_i \big(\partial_-^2 f \overline{\partial}^i g + \overline{\partial}^i f \partial_-^2 g \big) \bigg] + O(\theta^3).
\end{aligned}
\tag{3.29}
$$

3.3. Weyl ordering

The original Gutt product [36] is based on the 'Weyl ordering' prescription whereby all monomials in $U(\mathfrak{n})$ are completely symmetrized over all elements of $\mathfrak{n}$. On $\mathcal{N}^{\mathbb{C}}$ it is defined by

$$\Omega_\star\big(\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\big) = {}^\circ_\circ \mathrm{e}^{\mathrm{i}k^a \mathsf{X}_a} {}^\circ_\circ := \mathrm{e}^{\mathrm{i}k^a \mathsf{X}_a}. \tag{3.30}$$

While this ordering is usually thought of as the 'canonical' ordering for the construction of star products, in our case it turns out to be drastically more complicated than the other orderings. Nevertheless, we shall present here its explicit construction for the sake of completeness and for later comparisons.

It is an extremely arduous task to compute products of the group elements (3.30) directly from the Baker–Campbell–Hausdorff formula (3.13). Instead, we shall construct an isomorphism $\mathcal{G} : \overline{U(\mathfrak{n})^{\mathbb{C}}} \to \overline{U(\mathfrak{n})^{\mathbb{C}}}$ which sends the time-ordered product defined by (3.17) into the Weyl-ordered product defined by (3.30), i.e.

$$\mathcal{G} \circ \Omega_* = \Omega_\star. \tag{3.31}$$

Then by defining $\mathcal{G}_\Omega := \Omega_*^{-1} \circ \mathcal{G} \circ \Omega_\star$, the star product $\star$ associated with the Weyl ordering prescription (3.30) may be computed as

$$f \star g = \mathcal{G}_\Omega\big(\mathcal{G}_\Omega^{-1}(f) * \mathcal{G}_\Omega^{-1}(g)\big), \qquad f, g \in C^\infty(\mathfrak{n}^\vee). \tag{3.32}$$

Explicitly, if

$${}^*_* \mathrm{e}^{\mathrm{i}k^a \mathsf{X}_a} {}^*_* = \mathrm{e}^{\mathrm{i}G^a(\boldsymbol{k})\mathsf{X}_a} \tag{3.33}$$

for some function $\boldsymbol{G} = (G^a) : \mathbb{R}^6 \to \mathbb{R}^6$, then the isomorphism $\mathcal{G}_\Omega : C^\infty(\mathfrak{n}^\vee) \to C^\infty(\mathfrak{n}^\vee)$ may be represented as the invertible differential operator

$$\mathcal{G}_\Omega = \mathrm{e}^{\mathrm{i}\boldsymbol{x}\cdot[\boldsymbol{G}(-\mathrm{i}\boldsymbol{\partial})+\mathrm{i}\boldsymbol{\partial}]}. \tag{3.34}$$

This relation just reflects the fact that the time-ordered and Weyl-ordered star products, although not identical, simply represent different ordering prescriptions for the same algebra and are therefore (cohomologically) *equivalent*. We will elucidate this property more thoroughly in section 4. Thus, once the map (3.33) is known, the Weyl-ordered star product $\star$ can be computed in terms of the time-ordered star product $*$ of section 3.1.

The functions $G^a(\boldsymbol{k})$ appearing in (3.33) are readily calculable through the Baker–Campbell–Hausdorff formula. It is clear from (3.17) that the coefficient of the time translation generator $\mathsf{J} \in \mathfrak{n}$ is simply

$$G^j(j, t, \boldsymbol{p}^\pm) = j. \tag{3.35}$$

From (3.13) it is also clear that the only terms proportional to P^i_+ come from commutators of the form $[\mathsf{J}, [\vec{s}, [\mathsf{J}, \mathsf{P}^i_+]]\vec{s}]$, and gathering all terms we find

$$\begin{aligned}\textstyle\sum_{i=1,2} G^{p_i^+}(j, t, \boldsymbol{p}^\pm)\mathsf{P}^i_+ &= -\mathrm{i}\sum_{n=0}^{\infty}\frac{B_n}{n!}\big[\underbrace{\mathrm{i}j\mathsf{J}, \big[\ldots, \big[\mathrm{i}j\mathsf{J}}_{n}, \mathrm{i}p_i^+\mathsf{P}^i_+\big]\big]\ldots\big] \\ &= p_i^+\sum_{n=0}^{\infty}\frac{B_n}{n!}(-\theta j)^n\mathsf{P}^i_+. \end{aligned}\tag{3.36}$$

Since $B_0 = 1$, $B_1 = -\frac{1}{2}$ and $B_{2k+1} = 0$, $\forall k \geqslant 1$, from (3.14) we thereby find

$$G^{p^+}(j, t, \boldsymbol{p}^\pm) = \frac{\boldsymbol{p}^+}{\phi_\theta(j)}, \tag{3.37}$$

where we have introduced the function

$$\phi_\theta(j) = \frac{1 - \mathrm{e}^{-\theta j}}{\theta j} \tag{3.38}$$

obeying the identities

$$\phi_\theta(j)\,\mathrm{e}^{\theta j} = \phi_{-\theta}(j), \qquad \phi_\theta(j)\phi_{-\theta}(j) = -\frac{2}{(\theta j)^2}(1 - \cos(\theta j)). \tag{3.39}$$

In a completely analogous way, one finds the coefficient of the P^i_- term to be given by

$$G^{p^-}(j, t, \boldsymbol{p}^\pm) = \frac{\boldsymbol{p}^-}{\phi_{-\theta}(j)}. \tag{3.40}$$

Finally, the non-vanishing contributions to the central element $\mathsf{T} \in \mathfrak{n}$ are given by

$$\begin{aligned} G^t(j, t, \boldsymbol{p}^\pm)\mathsf{T} = t\mathsf{T} - \mathrm{i}\sum_{n=1}^{\infty}\frac{B_{n+1}}{n!}\Big(\big[\mathrm{i}p_i^+\mathsf{P}^i_+, [\underbrace{\mathrm{i}j\mathsf{J}, \ldots [\mathrm{i}j\mathsf{J}}_{n}, \mathrm{i}p_i^-\mathsf{P}^i_-]\ldots]\big] \\ + \big[\mathrm{i}p_i^-\mathsf{P}^i_-, \big[\underbrace{\mathrm{i}j\mathsf{J}, \ldots \big[\mathrm{i}j\mathsf{J}}_{n}, \mathrm{i}p_i^+\mathsf{P}^i_+\big]\ldots\big]\big]\Big) \\ = t\mathsf{T} + 4\theta\boldsymbol{p}^+\cdot\boldsymbol{p}^-\sum_{n=1}^{\infty}\frac{B_{n+1}}{n!}(-\theta j)^n\mathsf{T}. \end{aligned}\tag{3.41}$$

By differentiating (3.36) and (3.38) with respect to $s = -\theta j$, we arrive finally at

$$G^t(j, t, \boldsymbol{p}^\pm) = t + 4\theta\boldsymbol{p}^+\cdot\boldsymbol{p}^-\gamma_\theta(j), \tag{3.42}$$

where we have introduced the function

$$\gamma_\theta(j) = \frac{1}{2} + \frac{(1 + \theta j)\,\mathrm{e}^{-\theta j} - 1}{(\mathrm{e}^{-\theta j} - 1)^2}. \tag{3.43}$$

From (3.34) we may now write the explicit form of the differential operator implementing the equivalence between the star products $*$ and $\star$ as

$$\begin{aligned}\mathcal{G}_\Omega = \exp\Bigg[&-2\mathrm{i}\theta x^+ \overline{\boldsymbol{\partial}} \cdot \boldsymbol{\partial}\left(1 + \frac{2(1-\mathrm{i}\theta\partial_-)\,\mathrm{e}^{\mathrm{i}\theta\partial_-} - 1}{(\mathrm{e}^{\mathrm{i}\theta\partial_-}-1)^2}\right) \\ &+ \overline{\boldsymbol{z}} \cdot \boldsymbol{\partial}\left(\frac{\mathrm{i}\theta\partial_-}{\mathrm{e}^{\mathrm{i}\theta\partial_-}-1} - 1\right) - \boldsymbol{z} \cdot \overline{\boldsymbol{\partial}}\left(\frac{\mathrm{i}\theta\partial_-}{\mathrm{e}^{-\mathrm{i}\theta\partial_-}-1} + 1\right)\Bigg].\end{aligned} \tag{3.44}$$

From (3.21) and (3.33) we may readily compute the products of Weyl symbols with the result

$$\begin{aligned}{}^{\circ}_{\circ}{}^{\circ}_{\circ}\mathrm{e}^{\mathrm{i}k^a \mathsf{X}_a}{}^{\circ}_{\circ} \cdot {}^{\circ}_{\circ}\mathrm{e}^{\mathrm{i}k'^a \mathsf{X}_a}{}^{\circ}_{\circ}{}^{\circ}_{\circ} = \exp\mathrm{i}\Bigg\{&\frac{\phi_\theta(j)p_i^+ + \mathrm{e}^{-\theta j}\phi_\theta(j')p_i'^+}{\phi_\theta(j+j')}\mathsf{P}_+^i + \frac{\phi_{-\theta}(j)p_i^- + \mathrm{e}^{\theta j}\phi_{-\theta}(j')p_i'^-}{\phi_{-\theta}(j+j')}\mathsf{P}_-^i \\ &+ (j+j')\mathsf{J} + [t + t' + \theta(\phi_{-\theta}(j)\phi_{-\theta}(j')\boldsymbol{p}^+ \cdot \boldsymbol{p}'^- - \phi_\theta(j)\phi_\theta(j')\boldsymbol{p}^- \cdot \boldsymbol{p}'^+) \\ &- 4\theta(\gamma_\theta(j+j')(\phi_{-\theta}(j)\boldsymbol{p}^+ + \mathrm{e}^{\theta j}\phi_{-\theta}(j')\boldsymbol{p}'^+) \cdot (\phi_\theta(j)\boldsymbol{p}^- + \mathrm{e}^{-\theta j}\phi_\theta(j')\boldsymbol{p}'^-) \\ &- \gamma_\theta(j)\phi_\theta(j)\phi_{-\theta}(j)\boldsymbol{p}^+ \cdot \boldsymbol{p}^- - \gamma_\theta(j')\phi_\theta(j')\phi_{-\theta}(j')\boldsymbol{p}'^+ \cdot \boldsymbol{p}'^-)]\mathsf{T}\Bigg\}.\end{aligned} \tag{3.45}$$

From (3.11) we may now compute the star products $\star$ between the coordinate functions on $\mathfrak{n}^\vee$ to be

$$\begin{aligned}
&x_a \star x_a = (x_a)^2, && x_a \star x^+ = x^+ \star x_a = x_a x^+, \\
&z_1 \star z_2 = z_2 \star z_1 = z_1 z_2, && \overline{z}_1 \star \overline{z}_2 = \overline{z}_2 \star \overline{z}_1 = \overline{z}_1 \overline{z}_2, \\
&x^- \star z_i = x^- z_i - \frac{\mathrm{i}}{2}\theta z_i, && z_i \star x^- = x^- z_i + \frac{\mathrm{i}}{2}\theta z_i, \\
&x^- \star \overline{z}_i = x^- \overline{z}_i + \frac{\mathrm{i}}{2}\theta \overline{z}_i, && \overline{z}_i \star x^- = x^- \overline{z}_i - \frac{\mathrm{i}}{2}\theta \overline{z}_i, \\
&z_i \star \overline{z}_i = z_i \overline{z}_i - \mathrm{i}\theta x^+, && \overline{z}_i \star z_i = z_i \overline{z}_i + \mathrm{i}\theta x^+.
\end{aligned} \tag{3.46}$$

These products are identical to those of the symmetric time ordering prescription (3.27). After some computation, from (3.9) and (3.10) we find for generic functions $f, g \in C^\infty(\mathfrak{n}^\vee)$ the formula

$$\begin{aligned}f \star g = \mu \circ \exp\Bigg\{&\theta x^+ \Bigg[\frac{1\otimes 1 + (\mathrm{i}\theta(\partial_- \otimes 1 + 1 \otimes \partial_-) - 1 \otimes 1)\mathrm{e}^{\mathrm{i}\theta\partial_-} \otimes \mathrm{e}^{\mathrm{i}\theta\partial_-}}{(\mathrm{e}^{\mathrm{i}\theta\partial_-} \otimes \mathrm{e}^{\mathrm{i}\theta\partial_-} - 1 \otimes 1)^2} \\ &\times \left(\frac{4\boldsymbol{\partial}^\top(\mathrm{e}^{-\mathrm{i}\theta\partial_-}-1)}{\theta\partial_-} \otimes \frac{\overline{\boldsymbol{\partial}}(\mathrm{e}^{-\mathrm{i}\theta\partial_-}-1)}{\theta\partial_-} - \frac{3\overline{\boldsymbol{\partial}}^\top(\mathrm{e}^{\mathrm{i}\theta\partial_-}-1)}{\theta\partial_-} \otimes \frac{\boldsymbol{\partial}(\mathrm{e}^{\mathrm{i}\theta\partial_-}-1)}{\theta\partial_-}\right. \\ &+ \left.\frac{4\overline{\boldsymbol{\partial}} \cdot \boldsymbol{\partial}\sin^2\left(\frac{\theta}{2}\partial_-\right)}{\theta^2\partial_-^2} \otimes 1 - 1 \otimes \frac{4\overline{\boldsymbol{\partial}} \cdot \boldsymbol{\partial}\sin^2\left(\frac{\theta}{2}\partial_-\right)}{\theta^2\partial_-^2}\right) \\ &+ \frac{4\mathrm{i}\overline{\boldsymbol{\partial}} \cdot \boldsymbol{\partial}}{\theta\partial_-}\left(\frac{\mathrm{i}\sin^2\left(\frac{\theta}{2}\partial_-\right)}{\theta\partial_-(\mathrm{e}^{\mathrm{i}\theta\partial_-}-1)} - 1\right) \otimes 1 + 1 \otimes \frac{4\mathrm{i}\overline{\boldsymbol{\partial}} \cdot \boldsymbol{\partial}}{\theta\partial_-}\left(\frac{\mathrm{i}\sin^2\left(\frac{\theta}{2}\partial_-\right)}{\theta\partial_-(\mathrm{e}^{\mathrm{i}\theta\partial_-}-1)} - 1\right) \\ &+ \frac{3\overline{\boldsymbol{\partial}}^\top(\mathrm{e}^{\mathrm{i}\theta\partial_-}-1)}{\theta\partial_-} \otimes \frac{\boldsymbol{\partial}(\mathrm{e}^{\mathrm{i}\theta\partial_-}-1)}{\theta\partial_-} + \frac{\boldsymbol{\partial}^\top(\mathrm{e}^{-\mathrm{i}\theta\partial_-}-1)}{\theta\partial_-} \otimes \frac{\overline{\boldsymbol{\partial}}(\mathrm{e}^{-\mathrm{i}\theta\partial_-}-1)}{\theta\partial_-}\Bigg] \\ &+ \frac{\overline{z}_i}{1 \otimes \mathrm{e}^{-\mathrm{i}\theta\partial_-} - \mathrm{e}^{\mathrm{i}\theta\partial_-} \otimes 1}\Bigg[\frac{\partial^i}{\partial_-}(1 - \mathrm{e}^{\mathrm{i}\theta\partial_-}) \otimes \partial_-\partial_- \otimes \frac{\partial^i}{\partial_-}(1 - \mathrm{e}^{-\mathrm{i}\theta\partial_-}) \\ &+ 1 \otimes \partial^i \mathrm{e}^{-\mathrm{i}\theta\partial_-} - \partial^i \mathrm{e}^{\mathrm{i}\theta\partial_-} \otimes 1 - 1 \otimes 2\partial^i\Bigg]\end{aligned}$$

$$+\frac{z_i}{1\otimes e^{i\theta\partial_-}-e^{-i\theta\partial_-}\otimes 1}\left[\frac{\overline{\partial}^i}{\partial_-}(1-e^{-i\theta\partial_-})\otimes\partial_-\partial_-\otimes\frac{\overline{\partial}^i}{\partial_-}(1-e^{i\theta\partial_-})\right.$$
$$\left.\left.+1\otimes\overline{\partial}^i e^{i\theta\partial_-}-\overline{\partial}^i e^{-i\theta\partial_-}\otimes 1-1\otimes 2\overline{\partial}^i\right]\right\}f\otimes g. \tag{3.47}$$

To second order in the deformation parameter θ, we obtain

$$\begin{aligned}
f\star g = fg &- \frac{i}{2}\theta[2x^+(\overline{\boldsymbol{\partial}}f\cdot\boldsymbol{\partial}g-\boldsymbol{\partial}f\cdot\overline{\boldsymbol{\partial}}g)\\
&+\overline{\boldsymbol{z}}\cdot(\boldsymbol{\partial}f\partial_-g-\partial_-f\boldsymbol{\partial}g)+\boldsymbol{z}\cdot(\partial_-f\overline{\boldsymbol{\partial}}g-\overline{\boldsymbol{\partial}}f\partial_-g)]\\
&-\frac{1}{2}\theta^2\sum_{i=1,2}\left[(x^+)^2((\overline{\partial}^i)^2f(\partial^i)^2g+(\partial^i)^2f(\overline{\partial}^i)^2g-2\overline{\partial}^i\partial^i f\overline{\partial}^i\partial^i g)\right.\\
&-\frac{1}{3}x^+(\partial^i f\overline{\partial}^i\partial_-g+\overline{\partial}^i f\partial^i\partial_-g+\overline{\partial}^i\partial_-f\partial^i g+\partial^i\partial_-f\overline{\partial}^i g\\
&-2\partial_-f\overline{\partial}^i\partial^i g-2\overline{\partial}^i\partial^i f\partial_-g)\\
&+x^+\overline{z}_i(\overline{\partial}^i\partial^i f\partial^i\partial_-g-\overline{\partial}^i\partial_-f(\partial^i)^2g+\partial^i\partial_-f\overline{\partial}^i\partial^i g-(\partial^i)^2f\overline{\partial}^i\partial_-g)\\
&+x^+z_i(\overline{\partial}^i\partial^i f\overline{\partial}^i\partial_-g-\partial^i\partial_-f(\overline{\partial}^i)^2g+\overline{\partial}^i\partial_-f\overline{\partial}^i\partial^i g-(\overline{\partial}^i)^2f\partial^i\partial_-g)\\
&+\frac{1}{2}\overline{z}_iz_i(\overline{\partial}^i\partial_-f\partial^i\partial_-g+\partial^i\partial_-f\overline{\partial}^i\partial_-g-\partial_-^2f\overline{\partial}^i\partial^i g-\overline{\partial}^i\partial^i f\partial_-^2g)\\
&+\frac{1}{4}\overline{z}_i^2((\partial^i)^2f\partial_-^2g-2\partial^i\partial_-f\partial^i\partial_-g+\partial_-^2f(\partial^i)^2g)\\
&+\frac{1}{4}z_i^2((\overline{\partial}^i)^2f\partial_-^2g-2\overline{\partial}^i\partial-f\overline{\partial}^i\partial_-g+\partial_-^2f(\overline{\partial}^i)^2g)\\
&+\frac{1}{6}\overline{z}_i(\partial^i f\partial_-^2g+\partial_-^2f\partial^i g-\partial_-f\partial^i\partial_-g-\partial^i\partial_-f\partial_-g)\\
&\left.+\frac{1}{6}z_i(\partial_-^2f\overline{\partial}^i g+\overline{\partial}^i f\partial_-^2g-\partial_-f\overline{\partial}^i\partial_-g-\overline{\partial}^i\partial_-f\partial_-g)\right]+O(\theta^3).
\end{aligned} \tag{3.48}$$

Although extremely cumbersome in form, the Weyl-ordered product has several desirable features over the simpler time-ordered products. For instance, the Schwartz subspace of $C^\infty(\mathfrak{n}^\vee)$ is closed under the Weyl-ordered product, whereas the other products are only formal in this regard and do not define strict deformation quantizations. It is also Hermitian owing to the property

$$\overline{f\star g}=\overline{g}\star\overline{f}. \tag{3.49}$$

Moreover, while the $\mathfrak{n}$-covariance condition (3.5) holds for all of our star products, the Weyl product is in fact $\mathfrak{n}$-invariant, because for any $x\in\mathbb{C}_{(1)}(\mathfrak{n}^\vee)$ one has the stronger compatibility condition

$$[x,f]_\star=i\theta\Theta(x,f),\qquad\forall f\in C^\infty(\mathfrak{n}^\vee) \tag{3.50}$$

with the action of the Lie algebra $\mathfrak{n}$. In the next section, we shall see that the Weyl-ordered star product is, in a certain sense, the generator of all other star products making it the 'universal' product for the quantization of the spacetime NW_6.

4. Weyl systems

In this section, we will use the notion of a generalized Weyl system introduced in [1] to describe some more formal aspects of the star products that we have constructed and to analyse the interplay between them. This generalizes the standard Weyl systems [62] which may be used to provide a purely operator theoretic characterization of the Moyal product, associated with the (untwisted) Heisenberg algebra. In that case, it can be regarded as a projective representation of the translation group in an even-dimensional real vector space. However, for the twisted Heisenberg algebra such a representation is not possible, since by definition the appropriate arena should be a central extension of the non-Abelian subgroup $\mathcal{S}$ of the full Euclidean group $ISO(4)$. This requires a generalization of the standard notion which we will now describe and use it to obtain a very useful characterization of the noncommutative geometry induced by the algebra $\mathfrak{n}$.

Let $\mathbb{V}$ be a five-dimensional real vector space. In a suitable (canonical) basis, vectors $\boldsymbol{k} \in \mathbb{V} \cong \mathbb{R} \times \mathbb{C}^2$ will be denoted (with respect to a chosen complex structure) as

$$\boldsymbol{k} = \begin{pmatrix} j \\ \boldsymbol{p}^+ \\ \boldsymbol{p}^- \end{pmatrix} \tag{4.1}$$

with $j \in \mathbb{R}$ and $\boldsymbol{p}^\pm = \overline{\boldsymbol{p}^\mp} \in \mathbb{C}^2$. As the notation suggests, we regard $\mathbb{V}$ as the 'momentum space' of the dual $\mathfrak{n}^\vee$. Note that we do not explicitly incorporate the component corresponding to the central element T, as it will instead appear through the appropriate projective representation that we will construct, similarly to the Moyal case. As an Abelian group, $\mathbb{V} \cong \mathbb{R}^5$ with the usual addition + and identity $\mathbf{0}$. Corresponding to a deformation parameter $\theta \in \mathbb{R}$, we deform this Abelian Lie group structure to a generically non-Abelian one. The deformed composition law is denoted as $\boxplus$. It is associative and in general will depend on θ. The identity element with respect to $\boxplus$ is still defined to be $\mathbf{0}$, and the inverse of any element $\boldsymbol{k} \in \mathbb{V}$ is denoted as $\underline{\boldsymbol{k}}$, so that

$$\boldsymbol{k} \boxplus \underline{\boldsymbol{k}} = \underline{\boldsymbol{k}} \boxplus \boldsymbol{k} = \mathbf{0}. \tag{4.2}$$

Being a deformation of the underlying Abelian group structure on $\mathbb{V}$ means that the composition of any two vectors $\boldsymbol{k}, \boldsymbol{q} \in \mathbb{V}$ has a formal small θ expansion of the form

$$\boldsymbol{k} \boxplus \boldsymbol{q} = \boldsymbol{k} + \boldsymbol{q} + O(\theta), \tag{4.3}$$

from which it follows that

$$\underline{\boldsymbol{k}} = -\boldsymbol{k} + O(\theta). \tag{4.4}$$

In other words, rather than introducing star products that deform the pointwise multiplication of functions on $\mathfrak{n}^\vee$, we now deform the 'momentum space' of $\mathfrak{n}^\vee$ to a non-Abelian Lie group. We will see below that the five-dimensional group $(\mathbb{V}, \boxplus)$ is isomorphic to the original subgroup $\mathcal{S} \subset ISO(4)$, and that the two notions of quantization are in fact the same.

Given such a group, we now define a (generalized) Weyl system for the algebra $\mathfrak{n}$ as a quadruple $(\mathbb{V}, \boxplus, \mathsf{W}, \omega)$, where the map

$$\mathsf{W} : \mathbb{V} \longrightarrow \overline{U(\mathfrak{n})^{\mathbb{C}}} \tag{4.5}$$

is a projective representation of the group $(\mathbb{V}, \boxplus)$ with projective phase $\omega : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$. This means that for every pair of elements $\boldsymbol{k}, \boldsymbol{q} \in \mathbb{V}$ one has the composition rule

$$\mathsf{W}(\boldsymbol{k}) \cdot \mathsf{W}(\boldsymbol{q}) = \mathrm{e}^{\frac{\mathrm{i}}{2}\omega(\boldsymbol{k},\boldsymbol{q})\mathsf{T}} \cdot \mathsf{W}(\boldsymbol{k} \boxplus \boldsymbol{q}) \tag{4.6}$$

in the completed, complexified universal enveloping algebra of $\mathfrak{n}$. The associativity of $\boxplus$ and relation (4.6) imply that the subalgebra $\mathsf{W}(\mathbb{V}) \subset \overline{U(\mathfrak{n})^{\mathbb{C}}}$ is associative if and only if

$$\omega(\boldsymbol{k} \boxplus \boldsymbol{p}, \boldsymbol{q}) = \omega(\boldsymbol{k}, \boldsymbol{p} \boxplus \boldsymbol{q}) + \omega(\boldsymbol{p}, \boldsymbol{q}) - \omega(\boldsymbol{k}, \boldsymbol{p}) \tag{4.7}$$

for all vectors $\boldsymbol{k}, \boldsymbol{q}, \boldsymbol{p} \in \mathbb{V}$. This condition means that ω defines a 1-cocycle in the group cohomology of $(\mathbb{V}, \boxplus)$. It is automatically satisfied if ω is a bilinear form with respect to $\boxplus$.

We will in addition require that $\omega(\boldsymbol{k},\boldsymbol{q}) = O(\theta), \forall \boldsymbol{k},\boldsymbol{q} \in \mathbb{V}$ for consistency with (4.3). The identity element of $\mathsf{W}(\mathbb{V})$ is $\mathsf{W}(\mathbf{0})$ while the inverse of $\mathsf{W}(\boldsymbol{k})$ is given by

$$\mathsf{W}(\boldsymbol{k})^{-1} = \mathsf{W}(\underline{\boldsymbol{k}}). \tag{4.8}$$

The standard Weyl system on $\mathbb{R}^{2n}$ takes $\boxplus$ to be ordinary addition and ω to be the Darboux symplectic 2-form, so that $\mathsf{W}(\mathbb{R}^{2n})$ is a projective representation of the translation group, as is appropriate to the Moyal product.

Given a Weyl system defined as above, we can now introduce another isomorphism

$$\Pi : C^\infty(\mathbb{R}^5) \longrightarrow \mathsf{W}(\mathbb{V}) \tag{4.9}$$

defined by the symbol

$$\Pi(f) := \int_{\mathbb{R}^5} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^5} \tilde{f}(\boldsymbol{k}) \mathsf{W}(\boldsymbol{k}), \tag{4.10}$$

where as before $\tilde{f}$ denotes the Fourier transform of $f \in C^\infty(\mathbb{R}^5)$. This definition implies that

$$\Pi(\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}) = \mathsf{W}(\boldsymbol{k}), \tag{4.11}$$

and that we may introduce a $*$-involution $\dagger$ on both algebras $C^\infty(\mathbb{R}^5)$ and $\mathsf{W}(\mathbb{V})$ by the formula

$$\Pi(f^\dagger) = \Pi(f)^\dagger := \int_{\mathbb{R}^5} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^5} \overline{\tilde{f}(\underline{\boldsymbol{k}})} \mathsf{W}(\boldsymbol{k}). \tag{4.12}$$

The compatibility condition

$$(\Pi(f)\cdot\Pi(g))^\dagger = \Pi(g)^\dagger \cdot \Pi(f)^\dagger \tag{4.13}$$

with the product in $\overline{U(\mathfrak{n})^{\mathbb{C}}}$ imposes further constraints on the group composition law $\boxplus$ and cocycle ω [1]. From (4.6) we may thereby define a $\dagger$-Hermitian star product of $f, g \in C^\infty(\mathbb{R}^5)$ by the formula

$$f \star g := \Pi^{-1}(\Pi(f)\cdot\Pi(g)) = \int_{\mathbb{R}^5} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^5} \int_{\mathbb{R}^5} \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^5} \tilde{f}(\boldsymbol{k})\tilde{g}(\boldsymbol{q})\, \mathrm{e}^{\frac{\mathrm{i}}{2}\omega(\boldsymbol{k},\boldsymbol{q})} \Pi^{-1} \circ \mathsf{W}(\boldsymbol{k} \boxplus \boldsymbol{q}), \tag{4.14}$$

and in this way we have constructed a quantization of the algebra $\mathfrak{n}$ solely from the formal notion of a Weyl system. The associativity of $\star$ follows from associativity of $\boxplus$. We may also rewrite the star product (4.14) in terms of a bi-differential operator as

$$f \star g = f \exp\left(\frac{\mathrm{i}}{2}\omega(-\mathrm{i}\overleftarrow{\boldsymbol{\partial}}, -\mathrm{i}\overrightarrow{\boldsymbol{\partial}}) + \mathrm{i}\boldsymbol{x}\cdot(-\mathrm{i}\overleftarrow{\boldsymbol{\partial}} \boxplus -\mathrm{i}\overrightarrow{\boldsymbol{\partial}} + \mathrm{i}\overleftarrow{\boldsymbol{\partial}} + \mathrm{i}\overrightarrow{\boldsymbol{\partial}})\right) g. \tag{4.15}$$

This deformation is completely characterized in terms of the new algebraic structure and its projective representation provided by the Weyl system. It is straightforward to show that the Lie algebra of $(\mathbb{V}, \boxplus)$ coincides precisely with the original subalgebra $\mathfrak{s} \subset iso(4)$, while the cocycle ω generates the central extension of $\mathfrak{s}$ to $\mathfrak{n}$ in the usual way. From (4.14) one may compute the star products of coordinate functions on $\mathbb{R}^5$ as

$$x_a \star x_b = x_a x_b - \mathrm{i}\boldsymbol{x}\cdot \frac{\partial}{\partial k^a}\frac{\partial}{\partial q^b}(\boldsymbol{k}\boxplus\boldsymbol{q})\bigg|_{\boldsymbol{k}=\boldsymbol{q}=\mathbf{0}} - \frac{\mathrm{i}}{2}\frac{\partial}{\partial k^a}\frac{\partial}{\partial q^b}\omega(\boldsymbol{k},\boldsymbol{q})\bigg|_{\boldsymbol{k}=\boldsymbol{q}=\mathbf{0}}. \tag{4.16}$$

The corresponding star commutator may thereby be written as

$$[x_a, x_b]_\star = \mathrm{i}\theta C_{ab}^c x_c + \mathrm{i}\theta\xi_{ab}, \tag{4.17}$$

where the relation

$$\theta C_{ab}^c = -\left(\frac{\partial}{\partial k^a}\frac{\partial}{\partial q^b} - \frac{\partial}{\partial k^b}\frac{\partial}{\partial q^a}\right)(\boldsymbol{k}\boxplus\boldsymbol{q})^c\bigg|_{\boldsymbol{k}=\boldsymbol{q}=\mathbf{0}} \tag{4.18}$$

gives the structure constants of the Lie algebra defined by the Lie group $(\mathbb{V}, \boxplus)$, while the cocycle term

$$\theta\xi_{ab} = -\frac{1}{2}\left(\frac{\partial}{\partial k^a}\frac{\partial}{\partial q^b} - \frac{\partial}{\partial k^b}\frac{\partial}{\partial q^a}\right)\omega(\boldsymbol{k}, \boldsymbol{q})\bigg|_{\boldsymbol{k}=\boldsymbol{q}=\boldsymbol{0}} \tag{4.19}$$

gives the usual form of a central extension of this Lie algebra. Demanding that this yields a deformation quantization of the Kirillov–Kostant Poisson structure on $\mathfrak{n}^\vee$ requires that C_{ab}^c coincide with the structure constants of the subalgebra $\mathfrak{s} \subset iso(4)$ of $\mathfrak{n}$, and also that $\xi_{\boldsymbol{p}^-,\boldsymbol{p}^+} = -\xi_{\boldsymbol{p}^+,\boldsymbol{p}^-} = 2t$ be the only non-vanishing components of the central extension.

It is thus possible to define a broad class of deformation quantizations of $\mathfrak{n}^\vee$ solely in terms of an abstract Weyl system $(\mathbb{V}, \boxplus, \mathsf{W}, \omega)$, without explicit realization of the operators $\mathsf{W}(\boldsymbol{k})$. In the remainder of this section, we will set $\Pi = \Omega$ above and describe the Weyl systems underpinning the various products that we constructed previously. This entails identifying the appropriate maps (4.5), which enables the calculation of the projective representations (4.6) and hence explicit realizations of the group composition laws $\boxplus$ in the various instances. This unveils a purely algebraic description of the star products which will be particularly useful for our later constructions and enables one to make the equivalences between these products explicit.

4.1. Time ordering

Setting $t = t' = 0$ in (3.21), we find the 'time-ordered' non-Abelian group composition law $\circledast$ for any two elements of the form (4.1) to be given by

$$\boldsymbol{k} \circledast \boldsymbol{k}' = \begin{pmatrix} j + j' \\ \boldsymbol{p}^+ + \mathrm{e}^{-\theta j}\boldsymbol{p}'^+ \\ \boldsymbol{p}^- + \mathrm{e}^{\theta j}\boldsymbol{p}'^- \end{pmatrix}. \tag{4.20}$$

From (4.20) it is straightforward to compute the inverse $\underline{\boldsymbol{k}}$ of a group element (4.1), satisfying (4.2), to be

$$\underline{\boldsymbol{k}} = -\begin{pmatrix} j \\ \mathrm{e}^{\theta j}\boldsymbol{p}^+ \\ \mathrm{e}^{-\theta j}\boldsymbol{p}^- \end{pmatrix}. \tag{4.21}$$

The group cocycle is given by

$$\omega_*(\boldsymbol{k}, \boldsymbol{k}') = 2\mathrm{i}\theta(\mathrm{e}^{\theta j}\boldsymbol{p}^+ \cdot \boldsymbol{p}'^- - \mathrm{e}^{-\theta j}\boldsymbol{p}^- \cdot \boldsymbol{p}'^+) \tag{4.22}$$

and it defines the canonical symplectic structure on the j = constant subspaces $\mathbb{C}^2 \subset \mathbb{V}$. Note that in this representation the central coordinate function x^+ is not written explicitly and is simply understood as the unit element of $\mathbb{C}(\mathbb{R}^5)$, as is conventional in the case of the Moyal product. For $\boldsymbol{k} \in \mathbb{V}$ and $\mathsf{X}_a \in \mathfrak{s}$, the projective representation (4.6) is generated by the time-ordered group elements

$$\mathsf{W}_*(\boldsymbol{k}) = {}^*_*\,\mathrm{e}^{\mathrm{i}k^a\mathsf{X}_a}\,{}^*_* \tag{4.23}$$

defined in (3.16).

4.2. Symmetric time ordering

In a completely analogous manner, inspection of (3.26) reveals the 'symmetric time-ordered' non-Abelian group composition law $\boxdot$ defined by

$$\boldsymbol{k} \boxdot \boldsymbol{k}' = \begin{pmatrix} j + j' \\ \mathrm{e}^{\frac{\theta}{2}j'}\boldsymbol{p}^+ + \mathrm{e}^{-\frac{\theta}{2}j}\boldsymbol{p}'^+ \\ \mathrm{e}^{-\frac{\theta}{2}j'}\boldsymbol{p}^- + \mathrm{e}^{\frac{\theta}{2}j}\boldsymbol{p}'^- \end{pmatrix}, \tag{4.24}$$

for which the inverse $\underline{\boldsymbol{k}}$ of a group element (4.1) is simply given by

$$\underline{\boldsymbol{k}} = -\boldsymbol{k}. \tag{4.25}$$

The group cocycle is

$$\omega_\bullet(\boldsymbol{k}, \boldsymbol{k}') = 2\mathrm{i}\theta\left(\mathrm{e}^{\frac{\theta}{2}(j+j')}\boldsymbol{p}^+ \bullet \boldsymbol{p}'^- - \mathrm{e}^{-\frac{\theta}{2}(j+j')}\boldsymbol{p}^- \bullet \boldsymbol{p}'^+\right) \tag{4.26}$$

and it again induces the canonical symplectic structure on $\mathbb{C}^2 \subset \mathbb{V}$. The corresponding projective representation of $(\mathbb{V}, \boxdot)$ is generated by the symmetric time-ordered group elements

$$\mathsf{W}_\bullet(\boldsymbol{k}) = {}^\bullet_\bullet\, \mathrm{e}^{\mathrm{i}k^a\mathsf{X}_a} \,{}^\bullet_\bullet \tag{4.27}$$

defined in (3.25).

4.3. Weyl ordering

Finally, we construct the Weyl system $(\mathbb{V}, \boxed{\star}, \mathsf{W}_\star, \omega_\star)$ associated with the Weyl-ordered star product of section 3.3. Starting from (3.45) we introduce the non-Abelian group composition law $\boxed{\star}$ by

$$\boldsymbol{k} \,\boxed{\star}\, \boldsymbol{k}' = \begin{pmatrix} j + j' \\ \frac{\phi_\theta(j)\boldsymbol{p}^+ + \mathrm{e}^{-\theta j}\phi_\theta(j')\boldsymbol{p}'^+}{\phi_\theta(j+j')} \\ \frac{\phi_{-\theta}(j)\boldsymbol{p}^- + \mathrm{e}^{\theta j}\phi_{-\theta}(j')\boldsymbol{p}'^-}{\phi_{-\theta}(j+j')} \end{pmatrix}, \tag{4.28}$$

from which we may again straightforwardly compute the inverse $\underline{\boldsymbol{k}}$ of a group element (4.1) simply as

$$\underline{\boldsymbol{k}} = -\boldsymbol{k}. \tag{4.29}$$

When combined with definition (4.12), one has $f^\dagger = \overline{f}, \forall f \in C^\infty(\mathbb{R}^5)$ and this explains the Hermitian property (3.49) of the Weyl-ordered star product $\star$. This is also true of the product $\bullet$, whereas $*$ is only Hermitian with respect to the modified involution $\dagger$ defined by (4.12) and (4.21). The group cocycle is given by

$$\begin{aligned}\omega_\star(\boldsymbol{k}, \boldsymbol{k}') = -2\mathrm{i}\theta\big(&\phi_{-\theta}(j)\phi_{-\theta}(j')\boldsymbol{p}^+ \cdot \boldsymbol{p}'^- - \phi_\theta(j)\phi_\theta(j')\boldsymbol{p}^- \cdot \boldsymbol{p}'^+ \\ &- \gamma_\theta(j+j')(\phi_\theta(j)\boldsymbol{p}^+ + \mathrm{e}^{-\theta j}\phi_\theta(j')\boldsymbol{p}'^+) \cdot (\phi_{-\theta}(j)\boldsymbol{p}^- + \mathrm{e}^{\theta j}\phi_{-\theta}(j')\boldsymbol{p}'^-) \\ &+ \gamma_\theta(j)\phi_\theta(j)\phi_{-\theta}(j)\boldsymbol{p}^+ \cdot \boldsymbol{p}^- + \gamma_\theta(j')\phi_\theta(j')\phi_{-\theta}(j')\boldsymbol{p}'^+ \cdot \boldsymbol{p}'^-\big).\end{aligned} \tag{4.30}$$

In contrast to the other cocycles, this does *not* induce any symplectic structure, at least not in the manner described earlier. The corresponding projective representation (4.6) is generated by the completely symmetrized group elements

$$\mathsf{W}_\star(\boldsymbol{k}) = \mathrm{e}^{\mathrm{i}k^a\mathsf{X}_a} \tag{4.31}$$

with $\boldsymbol{k} \in \mathbb{V}$ and $\mathsf{X}_a \in \mathfrak{s}$.

The Weyl system $(\mathbb{V}, \boxed{\star}, \mathsf{W}_\star, \omega_\star)$ can be used to generate the other Weyl systems that we have found [1]. From (3.33) and (3.45) one has the identity

$$\mathsf{W}_*(j, \boldsymbol{p}^\pm) = \Omega_\star(\mathrm{e}^{\mathrm{i}(\boldsymbol{p}^+\cdot\overline{\boldsymbol{z}}+\boldsymbol{p}^-\cdot\boldsymbol{z})} \star \mathrm{e}^{\mathrm{i}jx^-}) \tag{4.32}$$

which implies that the time-ordered star product $*$ can be expressed by means of a choice of different Weyl system generating the product $\star$. Since $\Omega_\star$ is an algebra isomorphism, one has

$$\mathsf{W}_*(j, \boldsymbol{p}^\pm) = \mathsf{W}_\star(0, \boldsymbol{p}^\pm) \cdot \mathsf{W}_\star(j, \boldsymbol{0}). \tag{4.33}$$

This explicit relationship between the Weyl systems for the star products $*$ and $\star$ is another formulation of the statement of their cohomological equivalence, as established by other means in section 3.3. Similarly, the symmetric time-ordered star product $\bullet$ can be expressed in terms of $\star$ through the identity

$$\mathsf{W}_\bullet(j, \boldsymbol{p}^\pm) = \Omega_\star\left(\mathrm{e}^{\frac{\mathrm{i}}{2} j x^-} \star \mathrm{e}^{\mathrm{i}(\boldsymbol{p}^+\cdot\bar{\boldsymbol{z}}+\boldsymbol{p}^-\cdot\boldsymbol{z})} \star \mathrm{e}^{\frac{\mathrm{i}}{2} j x^-}\right), \tag{4.34}$$

which implies the relationship

$$\mathsf{W}_\bullet(j, \boldsymbol{p}^\pm) = \mathsf{W}_\star\left(\frac{j}{2}, \boldsymbol{0}\right) \cdot \mathsf{W}_\star(0, \boldsymbol{p}^\pm) \cdot \mathsf{W}_\star\left(\frac{j}{2}, \boldsymbol{0}\right) \tag{4.35}$$

between the corresponding Weyl systems. This shows explicitly that the star products $\bullet$ and $\star$ are also equivalent.

5. Twisted isometries

We will now start working our way towards the explicit construction of the geometric quantities required to define field theories on the noncommutative plane wave NW_6. We will begin with a systematic construction of derivative operators on the present noncommutative geometry, which will be used later on to write kinetic terms for scalar field actions. In this section, we will study some of the basic spacetime symmetries of the star products that we constructed in section 3, as they are directly related to the actions of derivations on the noncommutative algebras of functions.

Classically, the isometry group of the gravitational wave NW_6 is the group $\mathcal{N}_{\mathrm{L}} \times \mathcal{N}_{\mathrm{R}}$ induced by the left and right regular actions of the Lie group $\mathcal{N}$ on itself. The corresponding Killing vectors live in the 11-dimensional Lie algebra $\mathfrak{g} := \mathfrak{n}_{\mathrm{L}} \oplus \mathfrak{n}_{\mathrm{R}}$ (the left and right actions generated by the central element T coincide). This isometry group contains an $SO(4)$ subgroup acting by rotations in the transverse space $\boldsymbol{z} \in \mathbb{C}^2 \cong \mathbb{R}^4$, which is broken to $U(2)$ by the Neveu–Schwarz background (2.7). This symmetry can be restored upon quantization by instead letting the generators of $\mathfrak{g}$ act in a twisted fashion [21, 22, 64], as we now proceed to describe.

The action of an element $\nabla \in U(\mathfrak{g})$ as an algebra automorphism $C^\infty(\mathfrak{n}^\vee) \to C^\infty(\mathfrak{n}^\vee)$ will be denoted as $f \mapsto \nabla \triangleright f$. The universal enveloping algebra $U(\mathfrak{g})$ is given the structure of a cocommutative bialgebra by introducing the 'trivial' coproduct $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ defined by the homomorphism

$$\Delta(\nabla) = \nabla \otimes 1 + 1 \otimes \nabla, \tag{5.1}$$

which generates the action of $U(\mathfrak{g})$ on the tensor product $C^\infty(\mathfrak{n}^\vee) \otimes C^\infty(\mathfrak{n}^\vee)$. Since ∇ is an automorphism of $C^\infty(\mathfrak{n}^\vee)$, the action of the coproduct is compatible with the pointwise (commutative) product of functions $\mu : C^\infty(\mathfrak{n}^\vee) \otimes C^\infty(\mathfrak{n}^\vee) \to C^\infty(\mathfrak{n}^\vee)$ in the sense that

$$\nabla \triangleright \mu(f \otimes g) = \mu \circ \Delta(\nabla) \triangleright (f \otimes g). \tag{5.2}$$

For example, the standard action of spacetime translations is given by

$$\partial^a \triangleright f = \partial^a f \tag{5.3}$$

for which (5.2) becomes the classical symmetric Leibniz rule.

Let us now pass to a noncommutative deformation of the algebra of functions on NW_6 via a quantization map $\Omega : C^\infty(\mathfrak{n}^\vee) \to \overline{U(\mathfrak{n})^{\mathbb{C}}}$ corresponding to a specific star product $\star$ on $C^\infty(\mathfrak{n}^\vee)$ (or equivalently a specific operator ordering in $U(\mathfrak{n})$). This isomorphism can be used to induce an action of $U(\mathfrak{g})$ on the algebra $\overline{U(\mathfrak{n})^{\mathbb{C}}}$ through

$$\Omega(\nabla_\star) \triangleright \Omega(f) := \Omega(\nabla \triangleright f), \tag{5.4}$$

which defines a set of quantized operators $\nabla_\star = \nabla + O(\theta) : C^\infty(\mathfrak{n}^\vee) \to C^\infty(\mathfrak{n}^\vee)$. However, the bialgebra $U(\mathfrak{g})$ will no longer generate automorphisms with respect to the noncommutative star product on $C^\infty(\mathfrak{n}^\vee)$. It will only do so if its coproduct can be deformed to a non-cocommutative one $\Delta_\star = \Delta + O(\theta)$ such that the covariance condition

$$\nabla_\star \triangleright \mu_\star(f \otimes g) = \mu_\star \circ \Delta_\star(\nabla_\star) \triangleright (f \otimes g) \tag{5.5}$$

is satisfied, where $\mu_\star(f \otimes g) := f \star g$. This deformation is constructed by writing the star product $f \star g = \hat{\mathcal{D}}(f, g)$ in terms of a bi-differential operator as in (3.10) or (4.15) to define an invertible Abelian Drinfeld twist element [55] $\hat{\mathcal{F}}_\star \in \overline{U(\mathfrak{g})^{\mathbb{C}}} \otimes \overline{U(\mathfrak{g})^{\mathbb{C}}}$ through

$$f \star g = \mu \circ \hat{\mathcal{F}}_\star^{-1} \triangleright (f \otimes g). \tag{5.6}$$

It obeys the cocycle condition

$$(\hat{\mathcal{F}}_\star \otimes 1)(\Delta \otimes 1)\hat{\mathcal{F}}_\star = (1 \otimes \hat{\mathcal{F}}_\star)(\Delta \otimes 1)\hat{\mathcal{F}}_\star \tag{5.7}$$

and defines the twisted coproduct through

$$\Delta_\star := \hat{\mathcal{F}}_\star \circ \Delta \circ \hat{\mathcal{F}}_\star^{-1}, \tag{5.8}$$

where $(f \otimes g) \circ (f' \otimes g') := f f' \otimes g g'$. This new coproduct obeys the requisite coassociativity condition $(\Delta_\star \otimes \mathbb{1}) \circ \Delta_\star = (\mathbb{1} \otimes \Delta_\star) \circ \Delta_\star$. The important property of the twist element $\hat{\mathcal{F}}_\star$ is that it modifies only the coproduct on the bialgebra $U(\mathfrak{g})$, while leaving the original product structure (inherited from the Lie algebra $\mathfrak{g} = \mathfrak{n}_{\mathrm{L}} \oplus \mathfrak{n}_{\mathrm{R}}$) unchanged.

As an example, let us illustrate how to compute the twisting of the quantized translation generators by the noncommutative geometry of NW_6. For this, we introduce a Weyl system $(\mathbb{V}, \boxplus, \mathsf{W}, \omega)$ corresponding to the chosen star product $\star$. With the same notations as in the previous section, for $a = 1, \ldots, 5$ we may use (4.6) and (4.12) with $\Pi = \Omega$ and (5.4) with $\nabla = \partial^a$ to compute

$$\begin{aligned} \Omega(\partial^a_\star) \triangleright \Omega(\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}) \cdot \Omega(\mathrm{e}^{\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{x}}) &= \Omega(\partial^a_\star) \triangleright \mathrm{e}^{\frac{\mathrm{i}}{2}\omega(\boldsymbol{k},\boldsymbol{k}')\mathrm{T}} \cdot \Omega(\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\boxplus\boldsymbol{k}')\cdot\boldsymbol{x}}) \\ &= \mathrm{i}\,\mathrm{e}^{\frac{\mathrm{i}}{2}\omega(\boldsymbol{k},\boldsymbol{k}')\mathrm{T}} \cdot \Omega((\boldsymbol{k} \boxplus \boldsymbol{k}')^a\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\boxplus\boldsymbol{k}')\cdot\boldsymbol{x}}) \\ &= \mathrm{i}\sum_i \Omega\big(d^a_{(1)i}(-\mathrm{i}\boldsymbol{\partial}_\star)\big) \triangleright \Omega(\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}) \cdot \Omega\big(d^a_{(2)i}(-\mathrm{i}\boldsymbol{\partial}_\star)\big) \triangleright \Omega(\mathrm{e}^{\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{x}}), \end{aligned} \tag{5.9}$$

where we have assumed that the group composition law of the Weyl system has an expansion of the form $(\boldsymbol{k} \boxplus \boldsymbol{k}')^a := \sum_i d^a_{(1)i}(\boldsymbol{k}) d^a_{(2)i}(\boldsymbol{k}')$. From the covariance condition (5.5), it then follows that the twisted coproduct assumes a Sweedler form

$$\Delta_\star\left(\partial^a_\star\right) = \mathrm{i}\sum_i d^a_{(1)i}(-\mathrm{i}\boldsymbol{\partial}_\star) \otimes d^a_{(2)i}(-\mathrm{i}\boldsymbol{\partial}_\star). \tag{5.10}$$

Analogously, if we assume that the group cocycle of the Weyl system admits an expansion of the form $\omega(\boldsymbol{k}, \boldsymbol{k}') := \sum_i w^i_{(1)}(\boldsymbol{k}) w^i_{(2)}(\boldsymbol{k}')$, then a similar calculation gives the twisted coproduct of the quantized plane wave time derivative as

$$\Delta_\star(\partial^\star_+) = \partial^\star_+ \otimes 1 + 1 \otimes \partial^\star_+ - \frac{1}{2}\sum_i w^i_{(1)}(-\mathrm{i}\boldsymbol{\partial}_\star) \otimes w^i_{(2)}(-\mathrm{i}\boldsymbol{\partial}_\star). \tag{5.11}$$

Note that now the corresponding Leibniz rules (5.5) are no longer the usual ones associated with the product $\star$ but are the deformed, generically non-symmetric ones given by

$$\begin{aligned} \partial^a_\star \triangleright (f \star g) &= \mathrm{i}\sum_i \big(d^a_{(1)i}(-\mathrm{i}\boldsymbol{\partial}_\star) \triangleright f\big) \star \big(d^a_{(2)i}(-\mathrm{i}\boldsymbol{\partial}_\star) \triangleright g\big), \\ \partial^\star_+ \triangleright (f \star g) &= (\partial^\star_+ \triangleright f) \star g + f \star (\partial^\star_+ \triangleright g) - \frac{1}{2}\sum_i \big(w^i_{(1)}(-\mathrm{i}\boldsymbol{\partial}_\star) \triangleright f\big) \star \big(w^i_{(2)}(-\mathrm{i}\boldsymbol{\partial}_\star) \triangleright g\big) \end{aligned} \tag{5.12}$$

arising from the twisting of the coproduct. Thus, these derivatives do *not* define derivations of the noncommutative algebra of functions, but rather implement the twisting of isometries of flat space appropriate to the plane wave geometry [10, 24, 38, 45].

In the language of quantum groups [23], the twisted isometry group of the spacetime NW_6 coincides with the quantum double of the cocommutative Hopf algebra $U(\mathfrak{n})$. The antipode $S_\star : U(\mathfrak{g}) \to U(\mathfrak{g})$ of the given non-cocommutative Hopf algebra structure on the bialgebra $U(\mathfrak{g})$ gives the dual action of the isometries of the noncommutative plane wave and provides the analogue of inversion of isometry group elements. This analogy is made precise by computing $S_\star$ from the group inverses $\underline{\boldsymbol{k}}$ of elements $\boldsymbol{k} \in \mathbb{V}$ of the corresponding Weyl system. Symbolically, one has $S_\star(\boldsymbol{\partial}_\star) = \underline{\boldsymbol{\partial}_\star}$. In particular, if $\underline{\boldsymbol{k}} = -\boldsymbol{k}$ (as in the case of our symmetric star products) then $S_\star(\partial^a_\star) = -\partial^a_\star$ and the action of the antipode is trivial. In all three instances, the counit $\varepsilon_\star : U(\mathfrak{g}) \to \mathbb{C}$ describes the action on the trivial representation as $\varepsilon_\star(\partial^a_\star) = 0$, and it obeys the compatibility condition

$$(\varepsilon_\star \otimes 1)\hat{\mathcal{F}}_\star = 1 = (1 \otimes \varepsilon_\star)\hat{\mathcal{F}}_\star \tag{5.13}$$

with the Drinfeld twist. In what follows, we will only require the underlying bialgebra structure of $U(\mathfrak{g})$. The compatibility condition (5.5) means that the action of $U(\mathfrak{g})$ on $C^\infty(\mathfrak{n}^\vee)$ defines quantum isometries of the noncommutative pp-wave, in that the star product is an intertwiner and the noncommutative algebra of functions is covariant with respect to the action of the quantum group.

The generic non-triviality of the twisted coproducts (5.10) and (5.11) is consistent with and extends the fact that generic translations are not classically isometries of the plane wave geometry, but rather only appropriate twisted versions are [10, 24, 38, 45]. Similar computations can also be carried through for the remaining five isometry generators of $\mathfrak{g}$ and correspond to the right-acting counterparts of the derivatives above, giving the full action of the noncommutative isometry group on NW_6. We shall not display these formulae here. In the next section, we will explicitly construct the quantized derivative operators $\partial^a_\star$ and $\partial^\star_+$ above. We now proceed to list the coproducts corresponding to our three star products.

5.1. Time ordering

The Drinfeld twist $\hat{\mathcal{F}}_*$ for the time-ordered star product is the inverse of the exponential operator appearing in (3.23). Following the general prescription given above, from the group composition law (4.20) of the corresponding Weyl system we deduce the time-ordered coproducts

$$\begin{aligned} \Delta_*(\partial^*_-) &= \partial^*_- \otimes 1 + 1 \otimes \partial^*_-, \\ \Delta_*(\partial^i_*) &= \partial^i_* \otimes 1 + \mathrm{e}^{\mathrm{i}\theta\partial^*_-} \otimes \partial^i_*, \\ \Delta_*(\overline{\partial}^i_*) &= \overline{\partial}^i_* \otimes 1 + \mathrm{e}^{-\mathrm{i}\theta\partial^*_-} \otimes \overline{\partial}^i_*, \end{aligned} \tag{5.14}$$

while from the group cocycle (4.22) we obtain

$$\Delta_*(\partial^*_+) = \partial^*_+ \otimes 1 + 1 \otimes \partial^*_+ + \theta\, \mathrm{e}^{-\mathrm{i}\theta\partial^*_-} \boldsymbol{\partial}_*{}^\top \otimes \overline{\boldsymbol{\partial}}_* - \theta\, \mathrm{e}^{\mathrm{i}\theta\partial^*_-} \overline{\boldsymbol{\partial}}_*{}^\top \otimes \boldsymbol{\partial}_*. \tag{5.15}$$

The corresponding Leibniz rules read

$$\begin{aligned} &\partial^*_- \triangleright (f * g) = (\partial^*_- \triangleright f) * g + f * (\partial^*_- \triangleright g), \\ &\partial^*_+ \triangleright (f * g) = (\partial^*_+ \triangleright f) * g + f * (\partial^*_+ \triangleright g) \\ &\qquad + \theta\big(\mathrm{e}^{-\mathrm{i}\theta\partial^*_-} \boldsymbol{\partial}_*{}^\top \triangleright f\big) * (\overline{\boldsymbol{\partial}}_* \triangleright g) - \theta\big(\mathrm{e}^{\mathrm{i}\theta\partial^*_-} \overline{\boldsymbol{\partial}}_*{}^\top \triangleright f\big) * (\boldsymbol{\partial}_* \triangleright g), \\ &\partial^i_* \triangleright (f * g) = (\partial^i_* \triangleright f) * g + \big(\mathrm{e}^{\mathrm{i}\theta\partial^*_-} \triangleright f\big) * (\partial^i_* \triangleright g), \\ &\overline{\partial}^i_* \triangleright (f * g) = (\overline{\partial}^i_* \triangleright f) * g + (\mathrm{e}^{-\mathrm{i}\theta\partial^*_-} \triangleright f) * (\overline{\partial}^i_* \triangleright g). \end{aligned} \tag{5.16}$$

5.2. Symmetric time ordering

The Drinfeld twist $\hat{\mathcal{F}}_\bullet$ associated with the symmetric time-ordered star product is given by the inverse of the exponential operator in (3.28). From the group composition law (4.24) of the corresponding Weyl system, we deduce the symmetric time-ordered coproducts

$$\begin{aligned}
&\Delta_\bullet(\partial_-^\bullet) = \partial_-^\bullet \otimes 1 + 1 \otimes \partial_-^\bullet,\\
&\Delta_\bullet(\partial_\bullet^i) = \partial_\bullet^i \otimes \mathrm{e}^{-\frac{\mathrm{i}\theta}{2}\partial_-^\bullet} + \mathrm{e}^{\frac{\mathrm{i}\theta}{2}\partial_-^\bullet} \otimes \partial_\bullet^i,\\
&\Delta_\bullet(\overline{\partial}_\bullet^i) = \overline{\partial}_\bullet^i \otimes \mathrm{e}^{\frac{\mathrm{i}\theta}{2}\partial_-^\bullet} + \mathrm{e}^{-\frac{\mathrm{i}\theta}{2}\partial_-^\bullet} \otimes \overline{\partial}_\bullet^i,
\end{aligned} \tag{5.17}$$

while from the group cocycle (4.26) we find

$$\Delta_\bullet(\partial_+^\bullet) = \partial_+^\bullet \otimes 1 + 1 \otimes \partial_+^\bullet + \theta\, \mathrm{e}^{-\frac{\mathrm{i}\theta}{2}\partial_-^\bullet} \boldsymbol{\partial}_\bullet{}^\top \otimes \mathrm{e}^{-\frac{\mathrm{i}\theta}{2}\partial_-^\bullet} \overline{\boldsymbol{\partial}}_\bullet - \theta\, \mathrm{e}^{\frac{\mathrm{i}\theta}{2}\partial_-^\bullet} \overline{\boldsymbol{\partial}}_\bullet{}^\top \otimes \mathrm{e}^{\frac{\mathrm{i}\theta}{2}\partial_-^\bullet} \boldsymbol{\partial}_\bullet. \tag{5.18}$$

The corresponding Leibniz rules are given by

$$\begin{aligned}
&\partial_-^\bullet \triangleright (f \bullet g) = (\partial_-^\bullet \triangleright f) \bullet g + f \bullet (\partial_-^\bullet \triangleright g),\\
&\partial_+^\bullet \triangleright (f \bullet g) = (\partial_+^\bullet \triangleright f) \bullet g + f \bullet (\partial_+^\bullet \triangleright g) + \theta\big(\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}\partial_-^\bullet} \boldsymbol{\partial}_\bullet{}^\top \triangleright f\big) \bullet \big(\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}\partial_-^\bullet} \overline{\boldsymbol{\partial}}_\bullet \triangleright g\big)\\
&\qquad\qquad - \theta\big(\mathrm{e}^{\frac{\mathrm{i}\theta}{2}\partial_-^\bullet} \overline{\boldsymbol{\partial}}_\bullet{}^\top \triangleright f\big) \bullet \big(\mathrm{e}^{\frac{\mathrm{i}\theta}{2}\partial_-^\bullet} \boldsymbol{\partial}_\bullet \triangleright g\big),\\
&\partial_\bullet^i \triangleright (f \bullet g) = (\partial_\bullet^i \triangleright f) \bullet \big(\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}\partial_-^\bullet} \triangleright g\big) + \big(\mathrm{e}^{\frac{\mathrm{i}\theta}{2}\partial_-^\bullet} \triangleright f\big) \bullet (\partial_\bullet^i \triangleright g),\\
&\overline{\partial}_\bullet^i \triangleright (f \bullet g) = (\overline{\partial}_\bullet^i \triangleright f) \bullet \big(\mathrm{e}^{\frac{\mathrm{i}\theta}{2}\partial_-^\bullet} \triangleright g\big) + \big(\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}\partial_-^\bullet} \triangleright f\big) \bullet (\overline{\partial}_\bullet^i \triangleright g).
\end{aligned} \tag{5.19}$$

5.3. Weyl ordering

Finally, for the Weyl-ordered star product (3.47) we read off the twist element $\hat{\mathcal{F}}_\star$ in the standard way and use the associated group composition law (4.28) to write the coproducts

$$\begin{aligned}
&\Delta_\star(\partial_-^\star) = \partial_-^\star \otimes 1 + 1 \otimes \partial_-^\star,\\
&\Delta_\star(\partial_\star^i) = \frac{\phi_{-\theta}(\mathrm{i}\partial_-^\star)\partial_\star^i \otimes 1 + \mathrm{e}^{\mathrm{i}\theta\partial_-^\star} \otimes \phi_{-\theta}(\mathrm{i}\partial_-^\star)\partial_\star^i}{\phi_{-\theta}(\mathrm{i}\partial_-^\star \otimes 1 + 1 \otimes \mathrm{i}\partial_-^\star)},\\
&\Delta_\star(\overline{\partial}_\star^i) = \frac{\phi_{\theta}(\mathrm{i}\partial_-^\star)\overline{\partial}_\star^i \otimes 1 + \mathrm{e}^{-\mathrm{i}\theta\partial_-^\star} \otimes \phi_{\theta}(\mathrm{i}\partial_-^\star)\overline{\partial}_\star^i}{\phi_{\theta}(\mathrm{i}\partial_-^\star \otimes 1 + 1 \otimes \mathrm{i}\partial_-^\star)}.
\end{aligned} \tag{5.20}$$

The remaining coproduct may be determined from the cocycle (4.30) as

$$\begin{aligned}
\Delta_\star(\partial_+^\star) = {}& \partial_+^\star \otimes 1 + 1 \otimes \partial_+^\star + 2\mathrm{i}\theta\big[\phi_\theta(\mathrm{i}\partial_-^\star)\boldsymbol{\partial}_\star{}^\top \otimes \phi_\theta(\mathrm{i}\partial_-^\star)\overline{\boldsymbol{\partial}}_\star - \phi_{-\theta}(\mathrm{i}\partial_-^\star)\overline{\boldsymbol{\partial}}_\star{}^\top \otimes \phi_{-\theta}(\mathrm{i}\partial_-^\star)\boldsymbol{\partial}_\star\\
&+ (\gamma_\theta(\mathrm{i}\partial_-^\star) \otimes 1 - \gamma_\theta(\mathrm{i}\partial_-^\star \otimes 1 + 1 \otimes \mathrm{i}\partial_-^\star))(\phi_\theta(\mathrm{i}\partial_-^\star)\phi_{-\theta}(\mathrm{i}\partial_-^\star)\overline{\boldsymbol{\partial}}_\star \cdot \boldsymbol{\partial}_\star \otimes 1)\\
&+ (1 \otimes \gamma_\theta(\mathrm{i}\partial_-^\star) - \gamma_\theta(\mathrm{i}\partial_-^\star \otimes 1 + 1 \otimes \mathrm{i}\partial_-^\star))(1 \otimes \phi_\theta(\mathrm{i}\partial_-^\star)\phi_{-\theta}(\mathrm{i}\partial_-^\star)\overline{\boldsymbol{\partial}}_\star \cdot \boldsymbol{\partial}_\star)\\
&- \gamma_\theta(\mathrm{i}\partial_-^\star \otimes 1 + 1 \otimes \mathrm{i}\partial_-^\star)\big(\mathrm{e}^{-\mathrm{i}\theta\partial_-^\star}\phi_{-\theta}(\mathrm{i}\partial_-^\star)\boldsymbol{\partial}_\star{}^\top \otimes \phi_\theta(\mathrm{i}\partial_-^\star)\overline{\boldsymbol{\partial}}_\star\\
&+ \mathrm{e}^{\mathrm{i}\theta\partial_-^\star}\phi_\theta(\mathrm{i}\partial_-^\star)\overline{\boldsymbol{\partial}}_\star{}^\top \otimes \phi_{-\theta}(\mathrm{i}\partial_-^\star)\boldsymbol{\partial}_\star\big)\big].
\end{aligned} \tag{5.21}$$

In (5.20) and (5.21), the functionals of the derivative operator $\mathrm{i}\partial_-^\star \otimes 1 + 1 \otimes \mathrm{i}\partial_-^\star$ are understood as usual in terms of the power series expansions given in section 3.3. This leads to the corresponding Leibniz rules

$$\begin{aligned}
&\partial_-^\star \triangleright (f \star g) = (\partial_-^\star \triangleright f) \star g + f \star (\partial_-^\star \triangleright g),\\
&\partial_+^\star \triangleright (f \star g) = (\partial_+^\star \triangleright f) \star g + f \star (\partial_+^\star \triangleright g)\\
&\qquad + 2\mathrm{i}\theta\left\{\left(\frac{(1 - \mathrm{e}^{-\mathrm{i}\theta\partial_-^\star})\boldsymbol{\partial}_\star{}^\top}{\mathrm{i}\theta\partial_-^\star} \triangleright f\right) \star \left(\frac{(1 - \mathrm{e}^{-\mathrm{i}\theta\partial_-^\star})\overline{\boldsymbol{\partial}}_\star}{\mathrm{i}\theta\partial_-^\star} \triangleright g\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&- \left(\frac{(1-\mathrm{e}^{\mathrm{i}\theta\partial_-^\star})\overline{\boldsymbol{\partial}}_\star{}^\top}{\mathrm{i}\theta\partial_-^\star} \triangleright f \right) \star \left(\frac{(1-\mathrm{e}^{\mathrm{i}\theta\partial_-^\star})\boldsymbol{\partial}_\star}{\mathrm{i}\theta\partial_-^\star} \triangleright g \right) \\
&+ \left(\left[\frac{1}{2} + \frac{(1+\mathrm{i}\theta\partial_-^\star)\,\mathrm{e}^{-\mathrm{i}\theta\partial_-^\star}-1}{(\mathrm{e}^{-\mathrm{i}\theta\partial_-^\star}-1)^2} \right] \frac{\sin^2\left(\frac{\theta}{2}\partial_-^\star\right)\overline{\boldsymbol{\partial}}_\star\cdot\boldsymbol{\partial}_\star}{(\theta\partial_-^\star)^2} \triangleright f \right) \star g \\
&+ f \star \left(\left[\frac{1}{2} + \frac{(1+\mathrm{i}\theta\partial_-^\star)\,\mathrm{e}^{-\mathrm{i}\theta\partial_-^\star}-1}{(\mathrm{e}^{-\mathrm{i}\theta\partial_-^\star}-1)^2} \right] \frac{\sin^2\left(\frac{\theta}{2}\partial_-^\star\right)\overline{\boldsymbol{\partial}}_\star\cdot\boldsymbol{\partial}_\star}{(\theta\partial_-^\star)^2} \triangleright g \right) \\
&+ \sum_{n=1}^{\infty}\sum_{k=0}^{n} \frac{B_{n+1}(-\mathrm{i}\theta)^{n-2}}{k!(n-k)!} \left[\left((\partial_-^\star)^{n-k-2}\sin^2\left(\frac{\theta}{2}\partial_-^\star\right)\overline{\boldsymbol{\partial}}_\star\cdot\boldsymbol{\partial}_\star \triangleright f \right) \star ((\partial_-^\star)^k \triangleright g) \right. \\
&+ ((\partial_-^\star)^{n-k} \triangleright f) \star \left((\partial_-^\star)^{k-2}\sin^2\left(\frac{\theta}{2}\partial_-^\star\right)\overline{\boldsymbol{\partial}}_\star\cdot\boldsymbol{\partial}_\star \triangleright g \right) \\
&- ((\mathrm{e}^{-\mathrm{i}\theta\partial_-^\star}-1)(\partial_-^\star)^{n-k-1}\boldsymbol{\partial}_\star{}^\top \triangleright f) \star ((\mathrm{e}^{-\mathrm{i}\theta\partial_-^\star}-1)(\partial_-^\star)^{k-1}\overline{\boldsymbol{\partial}}_\star \triangleright g) \\
&\left. - ((\mathrm{e}^{\mathrm{i}\theta\partial_-^\star}-1)(\partial_-^\star)^{n-k-1}\overline{\boldsymbol{\partial}}_\star{}^\top \triangleright f) \star ((\mathrm{e}^{\mathrm{i}\theta\partial_-^\star}-1)(\partial_-^\star)^{k-1}\boldsymbol{\partial}_\star \triangleright g) \right],
\end{aligned}
$$

$$
\begin{aligned}
\partial_\star^i \triangleright (f\star g) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{B_n(\mathrm{i}\theta)^{n-1}}{k!(n-k)!} \big[((\mathrm{e}^{\mathrm{i}\theta\partial_-^\star}-1)(\partial_-^\star)^{n-k-1}\partial_\star^i \triangleright f) \star ((\partial_-^\star)^k \triangleright g) \\
&\quad + (\mathrm{e}^{\mathrm{i}\theta\partial_-^\star}(\partial_-^\star)^{n-k} \triangleright f) \star ((\mathrm{e}^{\mathrm{i}\theta\partial_-^\star}-1)(\partial_-^\star)^{k-1}\partial_\star^i \triangleright g) \big], \\
\overline{\partial}_\star^i \triangleright (f\star g) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{B_n(-\mathrm{i}\theta)^{n-1}}{k!(n-k)!} \big[((\mathrm{e}^{-\mathrm{i}\theta\partial_-^\star}-1)(\partial_-^\star)^{n-k-1}\overline{\partial}_\star^i \triangleright f) \star ((\partial_-^\star)^k \triangleright g) \\
&\quad + (\mathrm{e}^{-\mathrm{i}\theta\partial_-^\star}(\partial_-^\star)^{n-k} \triangleright f) \star ((\mathrm{e}^{-\mathrm{i}\theta\partial_-^\star}-1)(\partial_-^\star)^{k-1}\overline{\partial}_\star^i \triangleright g) \big].
\end{aligned}
\tag{5.22}
$$

Note that a common feature to all three deformations is that the coproduct of the quantization of the light-cone position translation generator ∂_- coincides with the trivial one (5.1), and thereby yields the standard symmetric Leibniz rule with respect to the pertinent star product. This owes to the fact that the action of ∂_- on the spacetime NW_6 corresponds to the commutative action of the central Lie algebra generator T, whose left and right actions coincide. In the next section, we shall see that the action of the quantized translations in x^- on $C^\infty(\mathfrak{n}^\vee)$ coincides with the standard commutative action (5.3). This is consistent with the fact that all frames of reference for the spacetime NW_6 possess an x^--translational symmetry, while translational symmetries in the other coordinates depend crucially on the frame and generally need to be twisted in order to generate an isometry of NW_6. Note also that ordinary time translation invariance is always broken by the time-dependent Neveu–Schwarz background (2.7).

6. Derivative operators

In this section, we will systematically construct a set of quantized derivative operators $\partial_\star^a, a = 1, \ldots, 6$, satisfying the conditions of the previous section. In general, there is no unique way to build up such derivatives. To this end, we will impose some weak conditions, namely that the quantized derivatives be deformations of ordinary derivatives, $\partial_\star^a = \partial^a + O(\theta)$, and that they commute among themselves, $[\partial_\star^a, \partial_\star^b]_\star = 0$. The latter condition is understood as a requirement for the iterated action of the derivatives on functions $f \in C^\infty(\mathfrak{n}^\vee)$, $[\partial_\star^a, \partial_\star^b]_\star \triangleright f = 0$ or equivalently

$$
\partial_\star^a \triangleright (\partial_\star^b \triangleright f) = \partial_\star^b \triangleright (\partial_\star^a \triangleright f). \tag{6.1}
$$

For the former condition, the simplest consistent choice is to assume a linear derivative deformation on the coordinate functions, $\left[\partial^a_\star, x_b\right]_\star = \delta^a_b + \mathrm{i}\theta\rho^a_{bc}\partial^c_\star$, which is understood as the requirement

$$[\partial^a_\star, x_b]_\star \triangleright f := \partial^a_\star \triangleright (x_b \star f) - x_b \star (\partial^a_\star \triangleright f) = \delta^a_b f + \mathrm{i}\theta\rho^a_{bc}\partial^c_\star \triangleright f. \tag{6.2}$$

A set of necessary conditions on the constant tensors $\rho^a_{bc} \in \mathbb{R}$ may be derived by demanding consistency of the derivatives with the original star commutators of coordinates (3.6). Applying the Jacobi identity for the star commutators between $\partial^a_\star$, x_b and x_c leads to the relations

$$\rho^a_{bc} - \rho^a_{cb} = C^a_{bc}, \qquad \rho^a_{bc}\rho^c_{de} - \rho^a_{dc}\rho^c_{be} = C^c_{bd}\rho^a_{ce}. \tag{6.3}$$

With these requirements we now seek to find quantized derivative operators $\partial^a_\star$ as functionals of ordinary derivatives ∂^a acting on $C^\infty(\mathfrak{n}^\vee)$ as in (5.3). However, there are (uncountably) infinitely many solutions ρ^a_{bc} obeying (6.3) [29] with C^c_{ab} the structure constants of the Lie algebra $\mathfrak{n}$ given by (2.1). We will choose the simplest consistent one defined by the star commutators

$$\begin{array}{llll}
[\partial^\star_-, x^-]_\star = 1, & [\partial^\star_+, x^-]_\star = 0, & [\partial^i_\star, x^-]_\star = -\mathrm{i}\theta\partial^i_\star, & [\overline{\partial}^i_\star, x^-]_\star = \mathrm{i}\theta\overline{\partial}^i_\star, \\
[\partial^\star_-, x^+]_\star = 0, & [\partial^\star_+, x^+]_\star = 1, & [\partial^i_\star, x^+]_\star = 0, & [\overline{\partial}^i_\star, x^+]_\star = 0, \\
[\partial^\star_-, z_i]_\star = 0, & [\partial^\star_+, z_i]_\star = -\mathrm{i}\theta\overline{\partial}^i_\star, & [\partial^i_\star, z_j]_\star = \delta^i_j, & [\overline{\partial}^i_\star, z_j]_\star = 0, \\
[\partial^\star_-, \overline{z}_i]_\star = 0, & [\partial^\star_+, \overline{z}_i]_\star = \mathrm{i}\theta\partial^i_\star, & [\partial^i_\star, \overline{z}_j]_\star = 0, & [\overline{\partial}^i_\star, \overline{z}_j]_\star = \delta^i_j,
\end{array} \tag{6.4}$$

whose $O(\theta)$ parts mimic the structure of the Lie brackets (2.1). This choice ensures that the derivatives $\partial^a_\star$ will generate the isometries appropriate to the quantization of the curved spacetime NW_6. All other admissible choices for ρ^a_{bc} can be mapped into those given by (6.4) via non-linear redefinitions of the derivative operators $\partial^a_\star$ [29]. It is important to realize that the quantized derivatives do not generally obey the classical Leibniz rule, i.e. $\partial^a_\star \triangleright (fg) \neq f(\partial^a_\star \triangleright g) + (\partial^a_\star \triangleright f)g$ in general, but rather the generalized Leibniz rules spelled out in the previous section in order to achieve consistency for $\theta \neq 0$. Let us now construct the three sets of derivatives of interest to us here.

6.1. Time ordering

For the time-ordered case, we use (3.23) to compute the star products

$$\begin{aligned} x^- * f &= (x^- - \mathrm{i}\theta \boldsymbol{z}\cdot\boldsymbol{\partial} + \mathrm{i}\theta\overline{\boldsymbol{z}}\cdot\overline{\boldsymbol{\partial}})f, \qquad & x^+ * f &= x^+ f, \\ z_i * f &= (z_i - \mathrm{i}\theta x^+\overline{\partial}^i)f, & \overline{z}_i * f &= (\overline{z}_i + \mathrm{i}\theta x^+\partial^i)f. \end{aligned} \tag{6.5}$$

Substituting these into (6.2) using (6.4) then shows that the actions of the $*$-derivatives simply coincide with the canonical actions of the translation generators on $C^\infty(\mathfrak{n}^\vee)$, so that

$$\partial^a_* \triangleright f = \partial^a f. \tag{6.6}$$

Thus, the time-ordered noncommutative geometry of NW_6 is invariant under *ordinary* translations of the spacetime in all coordinate directions, with the generators obeying the twisted Leibniz rules (5.16).

6.2. Symmetric time ordering

Next, consider the case of symmetric time ordering. From (3.28) we compute the star products

$$\begin{aligned} x^- \bullet f &= \left(x^- - \frac{\mathrm{i}\theta}{2}\boldsymbol{z}\cdot\boldsymbol{\partial} + \frac{\mathrm{i}\theta}{2}\overline{\boldsymbol{z}}\cdot\overline{\boldsymbol{\partial}}\right)f, \qquad & x^+ \bullet f &= x^+ f, \\ z_i \bullet f &= \mathrm{e}^{\frac{\mathrm{i}\theta}{2}\partial_-}(z_i - \mathrm{i}\theta x^+\overline{\partial}^i)f, & \overline{z}_i \bullet f &= \mathrm{e}^{-\frac{\mathrm{i}\theta}{2}\partial_-}(\overline{z}_i + \mathrm{i}\theta x^+\partial^i)f. \end{aligned} \tag{6.7}$$

Substituting (6.7) into (6.2) using (6.4) along with the derivative rule

$$\mathrm{e}^{\mathrm{i}\theta\partial_-}x^- = (x^- + \mathrm{i}\theta)\,\mathrm{e}^{\mathrm{i}\theta\partial_-}, \tag{6.8}$$

we find that the actions of the $\bullet$-derivatives on $C^\infty(\mathfrak{n}^\vee)$ are generically non-trivial and are given by

$$\begin{aligned} \partial_-^\bullet \triangleright f &= \partial_- f, & \partial_+^\bullet \triangleright f &= \partial_+ f, \\ \partial_\bullet^i \triangleright f &= \mathrm{e}^{-\frac{\mathrm{i}\theta}{2}\partial_-}\partial^i f, & \overline{\partial}_\bullet^i \triangleright f &= \mathrm{e}^{\frac{\mathrm{i}\theta}{2}\partial_-}\overline{\partial}^i f. \end{aligned} \tag{6.9}$$

Only the transverse space derivatives are modified owing to the fact that the Brinkman coordinate system is invariant under translations of the light-cone coordinates $x^\pm$. Again the twisted Leibniz rules (5.19) are straightforward to verify in this instance.

6.3. Weyl ordering

Finally, from the Weyl-ordered star product (3.47) we compute

$$\begin{aligned} x^- \star f &= \left[x^- + \left(1 - \frac{1}{\phi_{-\theta}(\mathrm{i}\partial_-)}\right)\frac{\boldsymbol{z}\cdot\boldsymbol{\partial}}{\partial_-} + \left(1 - \frac{1}{\phi_\theta(\mathrm{i}\partial_-)}\right)\frac{\overline{\boldsymbol{z}}\cdot\overline{\boldsymbol{\partial}}}{\partial_-} \right. \\ &\qquad \left. - 2\theta x^+\left(\frac{2}{\theta\partial_-} - \cot\left(\frac{\theta}{2}\partial_-\right)\right)\frac{\overline{\boldsymbol{\partial}}\cdot\boldsymbol{\partial}}{\partial_-}\right] f, \\ x^+ \star f &= x^+ f, \qquad z_i \star f = \left[\frac{z_i}{\phi_{-\theta}(\mathrm{i}\partial_-)} + 2x^+\left(1 - \frac{1}{\phi_{-\theta}(\mathrm{i}\partial_-)}\right)\frac{\overline{\partial}^i}{\partial_-}\right] f, \\ \overline{z}_i \star f &= \left[\frac{\overline{z}_i}{\phi_\theta(\mathrm{i}\partial_-)} + 2x^+\left(1 - \frac{1}{\phi_\theta(\mathrm{i}\partial_-)}\right)\frac{\partial^i}{\partial_-}\right] f. \end{aligned} \tag{6.10}$$

From (6.2), (6.4) and the derivative rule

$$\phi_\theta(\mathrm{i}\partial_-)x^- = \frac{\mathrm{e}^{\mathrm{i}\theta\partial_-} - \phi_\theta(\mathrm{i}\partial_-)}{\mathrm{i}\partial_-} + x^-\phi_\theta(\mathrm{i}\partial_-), \tag{6.11}$$

it then follows that the actions of the $\star$-derivatives on $C^\infty(\mathfrak{n}^\vee)$ are given by

$$\begin{aligned} \partial_-^\star \triangleright f &= \partial_- f, \\ \partial_+^\star \triangleright f &= \left[\partial_+ + 2\left(1 - \frac{\sin(\theta\partial_-)}{\theta\partial_-}\right)\frac{\overline{\boldsymbol{\partial}}\cdot\boldsymbol{\partial}}{\partial_-}\right] f, \\ \partial_\star^i \triangleright f &= -\frac{1 - \mathrm{e}^{\mathrm{i}\theta\partial_-}}{\mathrm{i}\theta\partial_-}\partial^i f, \\ \overline{\partial}_\star^i \triangleright f &= \frac{1 - \mathrm{e}^{-\mathrm{i}\theta\partial_-}}{\mathrm{i}\theta\partial_-}\overline{\partial}^i f. \end{aligned} \tag{6.12}$$

Thus, in the completely symmetric noncommutative geometry of NW_6 both the light-cone and the transverse space of the plane wave are generically only invariant under rather complicated twisted translations, obeying the involved Leibniz rules (5.22).

7. Traces

The final ingredient required to construct noncommutative field theory action functionals is a definition of integration. At the algebraic level, we define an integral to be a trace on the algebra $\overline{U(\mathfrak{n})^{\mathbb{C}}}$, i.e. a map $⨍ : \overline{U(\mathfrak{n})^{\mathbb{C}}} \to \mathbb{C}$ which is linear,

$$⨍\,(c_1\Omega(f) + c_2\Omega(g)) = c_1 ⨍\,\Omega(f) + c_2 ⨍\,\Omega(g) \tag{7.1}$$

for all $f, g \in C^\infty(\mathfrak{n}^\vee)$ and $c_1, c_2 \in \mathbb{C}$, and which is cyclic,

$$\int\!\!\!\!\!\!- \Omega(f)\cdot\Omega(g) = \int\!\!\!\!\!\!- \Omega(g)\cdot\Omega(f). \tag{7.2}$$

We define the integral in the star-product formalism using the usual definitions for the integration of commuting Schwartz functions in $C^\infty(\mathbb{R}^6)$. Then the linearity property (7.1) is automatically satisfied. To satisfy the cyclicity requirement (7.2), we introduce [2, 6, 17, 30, 34] a measure κ on $\mathbb{R}^6$ which deforms the flat space volume element $\mathrm{d}\boldsymbol{x}$ and define

$$\int\!\!\!\!\!\!- \Omega(f) := \int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\,\kappa(\boldsymbol{x})\,f(\boldsymbol{x}). \tag{7.3}$$

The measure κ is chosen in order to achieve the property (7.2), so that

$$\int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\,\kappa(\boldsymbol{x})(f\star g)(\boldsymbol{x}) = \int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\,\kappa(\boldsymbol{x})(g\star f)(\boldsymbol{x}). \tag{7.4}$$

Such a measure always exists [17, 30, 34] and its inclusion in the present context is natural for the curved spacetime NW_6 which we are considering here. It is important note that, for the star products that we use, a measure which satisfies (7.4) gives the integral the additional property

$$\int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\,\kappa(\boldsymbol{x})(f\star g)(\boldsymbol{x}) = \int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\,\kappa(\boldsymbol{x})\,f(\boldsymbol{x})g(\boldsymbol{x}), \tag{7.5}$$

providing an explicit realization of the Connes–Flato–Sternheimer conjecture [34].

Since the coordinate functions x_a generate the noncommutative algebra, the cyclicity constraint (7.4) is equivalent to the star-commutator condition

$$\int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\,\kappa(\boldsymbol{x})[(x_a)^n, f(\boldsymbol{x})]_\star = 0 \tag{7.6}$$

which must hold for arbitrary functions $f \in C^\infty(\mathbb{R}^6)$ (for which the integral makes sense) and for all $n \in \mathbb{N}$, $a = 1, \ldots, 6$. Expanding the star-commutator bracket using its derivation property brings (7.6) to the form

$$\int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\,\kappa(\boldsymbol{x}) \sum_{m=0}^{n} \binom{n}{m} (x_a)^{n-m} \star [x_a, f(\boldsymbol{x})]_\star \star (x_a)^m = 0. \tag{7.7}$$

We may thus insert the explicit form of $[x_a, f]_\star$ for generic f and use the ordinary integration by parts property

$$\int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x})g(\boldsymbol{x})(\partial^a)^n h(\boldsymbol{x}) = (-1)^n \int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}(f(\boldsymbol{x})(\partial^a)^n g(\boldsymbol{x})h(\boldsymbol{x}) + (\partial^a)^n f(\boldsymbol{x})g(\boldsymbol{x})h(\boldsymbol{x})) \tag{7.8}$$

for Schwartz functions $f, g, h \in C^\infty(\mathbb{R}^6)$. This will lead to a number of constraints on the measure κ.

The trace (7.3) can also be used to define an inner product $(-,-) : C^\infty(\mathfrak{n}^\vee) \times C^\infty(\mathfrak{n}^\vee) \to \mathbb{C}$ through

$$(f, g) := \int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\,\kappa(\boldsymbol{x})\big(\overline{f}\star g\big)(\boldsymbol{x}). \tag{7.9}$$

Note that this is different from the inner product introduced in section 2.1. When we come to deal with the variational principle in the next section, we shall require that our star-derivative operators $\partial^a_\star$ be anti-Hermitian with respect to the inner product (7.9), i.e. $(f, \partial^a_\star \triangleright g) = -(\partial^a_\star \triangleright f, g)$, or equivalently

$$\int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\,\kappa(\boldsymbol{x})(\overline{f}\star\partial^a_\star \triangleright g)(\boldsymbol{x}) = -\int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\,\kappa(\boldsymbol{x})(\overline{\partial^a_\star \triangleright f}\star g)(\boldsymbol{x}). \tag{7.10}$$

This allows for a generalized integration by parts property [30] for our noncommutative integral. As always, we will now go through our list of star products to explore the properties of the integral in each case. We will find that the measure κ is not uniquely determined by the above criteria and there is a large flexibility in the choices that can be made. We will also find that the derivatives of the previous section must be generically modified by a κ-dependent shift in order to satisfy (7.10).

7.1. Time ordering

Using (6.5) along with the analogous $*$-products $f * x_a$, we arrive at the $*$-commutators

$$\begin{aligned} &[x^-, f]_* = \mathrm{i}\theta(\overline{\boldsymbol{z}}\cdot\overline{\boldsymbol{\partial}} - \boldsymbol{z}\cdot\boldsymbol{\partial})f, \qquad [x^+, f]_* = 0, \\ &[z_i, f]_* = z_i(1 - \mathrm{e}^{-\mathrm{i}\theta\partial_-})f - \mathrm{i}\theta x^+(1 + \mathrm{e}^{-\mathrm{i}\theta\partial_-})\overline{\partial}^i f, \\ &[\overline{z}_i, f]_* = \overline{z}_i(1 - \mathrm{e}^{\mathrm{i}\theta\partial_-})f + \mathrm{i}\theta x^+(1 + \mathrm{e}^{\mathrm{i}\theta\partial_-})\partial^i f. \end{aligned} \tag{7.11}$$

When inserted into (7.7), after integration by parts and application of the derivative rule (6.8) these expressions imply constraints on the corresponding measure κ_* given by

$$\begin{aligned} &(1 - \mathrm{e}^{\mathrm{i}\theta\partial_-})\kappa_* = 0, \qquad (1 + \mathrm{e}^{\mathrm{i}\theta\partial_-})\overline{\partial}^i\kappa_* = 0, \\ &(1 - \mathrm{e}^{-\mathrm{i}\theta\partial_-})\partial^i\kappa_* = 0, \qquad \boldsymbol{z}\cdot\boldsymbol{\partial}\kappa_* = \overline{\boldsymbol{z}}\cdot\overline{\boldsymbol{\partial}}\kappa_*. \end{aligned} \tag{7.12}$$

It is straightforward to see that equations (7.12) imply that the measure must be independent of both the light-cone position and transverse coordinates, so that

$$\partial_-\kappa_* = \partial^i\kappa_* = \overline{\partial}^i\kappa_* = 0. \tag{7.13}$$

However, the derivative ∂_+^* in (6.6) does not satisfy the anti-Hermiticity requirement (7.10). This can be remedied by translating it by a logarithmic derivative of the measure κ_* and defining the modified $*$-derivative

$$\widetilde{\partial}_+^* = \partial_+ + \tfrac{1}{2}\partial_+ \ln\kappa_*. \tag{7.14}$$

The remaining $*$-derivatives in (6.6) are unaltered. While this redefinition has no adverse effects on the commutation relations (6.4), the action $\widetilde{\partial}_+^* \triangleright f$ contains an additional linear term in f even if the function f is independent of the time coordinate x^+.

7.2. Symmetric time ordering

Using (6.7) along with the corresponding $\bullet$-products $f \bullet x_a$, we arrive at the $\bullet$-commutators

$$\begin{aligned} &[x^-, f]_\bullet = \mathrm{i}\theta(\overline{\boldsymbol{z}}\cdot\overline{\boldsymbol{\partial}} - \boldsymbol{z}\cdot\boldsymbol{\partial})f, \qquad [x^+, f]_\bullet = 0, \\ &[z_i, f]_\bullet = 2\mathrm{i}z_i\sin\left(\frac{\theta}{2}\partial_-\right)f - 2\mathrm{i}\theta x^+\overline{\partial}^i\cos\left(\frac{\theta}{2}\partial_-\right)f, \\ &[\overline{z}_i, f]_\bullet = -2\mathrm{i}\overline{z}_i\sin\left(\frac{\theta}{2}\partial_-\right)f + 2\mathrm{i}\theta x^+\partial^i\cos\left(\frac{\theta}{2}\partial_-\right)f. \end{aligned} \tag{7.15}$$

Substituting these into (7.7) and integrating by parts, we arrive at constraints on the measure $\kappa_\bullet$ given by

$$(1 - \overline{\partial}^i)\sin\left(\frac{\theta}{2}\partial_-\right)\kappa_\bullet = 0, \qquad (1 + \partial^i)\sin\left(\frac{\theta}{2}\partial_-\right)\kappa_\bullet = 0, \qquad \boldsymbol{z}\cdot\boldsymbol{\partial}\kappa_\bullet = \overline{\boldsymbol{z}}\cdot\overline{\boldsymbol{\partial}}\kappa_\bullet \tag{7.16}$$

which can be reduced to the conditions

$$\boldsymbol{z}\cdot\boldsymbol{\partial}\kappa_\bullet = \overline{\boldsymbol{z}}\cdot\overline{\boldsymbol{\partial}}\kappa_\bullet, \qquad \partial_-\kappa_\bullet = 0. \tag{7.17}$$

Now the derivative operators $\partial_+^\bullet$, $\partial_\bullet^i$ and $\overline{\partial}_\bullet^i$ all violate the requirement (7.10). Introducing translates of $\partial_\bullet^i$ and $\overline{\partial}_\bullet^i$ analogously to what we did in (7.14) is problematic. While such a shift does not alter the canonical commutation relations between the coordinates and derivatives, i.e. the algebraic properties of the differential operators, it does violate the $\bullet$-commutator relationships (6.2) and (6.4) for generic functions f. Consistency between differential operator and function commutators would only be possible in this case by demanding that multiplication from the left follows a Leibniz-like rule for the translated part.

Thus, in order to satisfy both sets of constraints, we are forced to further require that the measure $\kappa_\bullet$ depends only on the plane wave time coordinate x^+ so that (7.17) truncates to

$$\partial^i\kappa_\bullet = \overline{\partial}^i\kappa_\bullet = \partial_-\kappa_\bullet = 0. \tag{7.18}$$

The logarithmic translation of $\partial_+^\bullet$ must still be applied in order to ensure that the time derivative is anti-Hermitian with respect to the noncommutative inner product. This modifies its action to

$$\widetilde{\partial}_+^\bullet = \partial_+ + \tfrac{1}{2}\partial_+ \ln\kappa_\bullet. \tag{7.19}$$

The actions of all other $\bullet$-derivatives are as in (6.9). Again this shifting has no adverse effects on (6.4), but it carries the same warning as in the time-ordered case regarding extra linear terms from the action $\widetilde{\partial}_+^\bullet \triangleright f$.

7.3. Weyl ordering

Finally, the Weyl-ordered star products (6.10) along with the corresponding $f \star x_a$ products lead to the $\star$-commutators

$$\begin{aligned}
&[x^-, f]_\star = \mathrm{i}\theta(\overline{\boldsymbol{z}}\cdot\overline{\boldsymbol{\partial}} - \boldsymbol{z}\cdot\boldsymbol{\partial})f, \qquad && [x^+, f]_\star = 0,\\
&[z_i, f]_\star = \mathrm{i}\theta(z_i\partial_- - 2x^+\overline{\partial}^i)f, \qquad && [\overline{z}_i, f]_\star = \mathrm{i}\theta(-\overline{z}_i\partial_- + 2x^+\partial^i)f.
\end{aligned} \tag{7.20}$$

Substituting these commutation relations into (7.7), integrating by parts and using the derivative rules (6.8) and (6.11) leads to the corresponding measure constraints

$$z_i\partial_-\kappa_\star = 2x^+\overline{\partial}^i\kappa_\star, \qquad \overline{z}_i\partial_-\kappa_\star = 2x^+\partial^i\kappa_\star, \qquad \boldsymbol{z}\cdot\boldsymbol{\partial}\kappa_\star = \overline{\boldsymbol{z}}\cdot\overline{\boldsymbol{\partial}}\kappa_\star. \tag{7.21}$$

Again these differential equations imply that the measure $\kappa_\star$ depends only on the plane wave time coordinate x^+ so that

$$\partial_-\kappa_\star = \partial^i\kappa_\star = \overline{\partial}^i\kappa_\star = 0. \tag{7.22}$$

Translating the derivative operator $\partial_+^\star$ as before in order to satisfy (7.10) yields the modified derivative

$$\widetilde{\partial}_+^\star = \partial_+ + 2\left(1 - \frac{\sin(\theta\partial_-)}{\theta\partial_-}\right)\frac{\overline{\boldsymbol{\partial}}\cdot\boldsymbol{\partial}}{\partial_-} + \frac{1}{2}\partial_+\ln\kappa_\star, \tag{7.23}$$

with the remaining $\star$-derivatives in (6.12) unchanged. Once again this produces no major alteration to (6.4) but does yield extra linear terms in the actions $\widetilde{\partial}_+^\star \triangleright f$.

8. Field theory on NW_6

We are now ready to apply the detailed constructions of the preceding sections to the analysis of noncommutative field theories on the plane wave NW_6, regarded as the worldvolume of a non-symmetric D5-brane [48]. In this paper, we will only study the simplest example of free scalar fields, leaving the detailed analysis of interacting field theories and higher spin (fermionic and gauge) fields for future work. The analysis of this section will set the stage for more detailed studies of noncommutative field theories in these settings and will illustrate some of the generic features that one can expect.

Given a real scalar field $\varphi \in C^\infty(\mathfrak{n}^\vee)$ of mass m, we define an action functional using the integral (7.3) by

$$S[\varphi] = \int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\,\kappa(\boldsymbol{x}) \left[\frac{1}{2}\eta_{ab}(\widetilde{\partial}^a_\star \triangleright \varphi) \star (\widetilde{\partial}^b_\star \triangleright \varphi) + \frac{1}{2}m^2 \varphi \star \varphi\right], \tag{8.1}$$

where η_{ab} is the invariant Minkowski metric tensor induced by the inner product (2.2) with the non-vanishing components $\eta_{\pm\mp} = 1$ and $\eta_{z_i \bar{z}_j} = \frac{1}{2}\delta_{ij}$. The tildes on the derivatives in (8.1) indicate that the time component must be appropriately shifted as described in the previous section. Using the property (7.5), we may simplify the action to the form

$$S[\varphi] = \int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\,\kappa(\boldsymbol{x}) \left[\frac{1}{2}\eta_{ab}(\widetilde{\partial}^a_\star \triangleright \varphi)(\widetilde{\partial}^b_\star \triangleright \varphi) + \frac{1}{2}m^2 \varphi^2\right]. \tag{8.2}$$

By using the integration by parts property (7.10) on Schwartz fields φ, we may easily compute the first-order variation of the action (8.2) to be

$$\frac{\delta S[\varphi]}{\delta\varphi}\delta\varphi := S[\varphi + \delta\varphi] - S[\varphi] = -\int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{x}\,\kappa(\boldsymbol{x})[\eta_{ab}\overline{\widetilde{\partial}^a_\star} \triangleright (\widetilde{\partial}^b_\star \triangleright \varphi) - m^2\varphi^2]\delta\varphi. \tag{8.3}$$

Applying the variational principle $\frac{\delta S[\varphi]}{\delta\varphi} = 0$ to (8.3) thereby leads to the noncommutative Klein–Gordan field equation

$$\square^\star \triangleright \varphi - m^2 \varphi = 0, \tag{8.4}$$

where

$$\square^\star \triangleright \varphi := 2\partial_+ \triangleright \partial_- \varphi + \boldsymbol{\partial}^\top \triangleright \overline{\boldsymbol{\partial}} \triangleright \varphi + \tfrac{1}{2}\partial_+ \ln\kappa\,\partial_-\varphi \tag{8.5}$$

and we have used $\partial_-\kappa = 0$. The second-order $\star$-differential operator $\square^\star$ should be regarded as a deformation of the covariant Laplace operator $\square^\star_0$ corresponding to the commutative plane wave geometry of NW_6. This Laplacian coincides with the quadratic Casimir element

$$\mathsf{C} := \theta^{-2}\eta^{ab}\mathsf{X}_a\mathsf{X}_b = 2\mathsf{J}\mathsf{T} + \frac{1}{2}\sum_{i=1,2}\left(\mathsf{P}^i_+\mathsf{P}^i_- + \mathsf{P}^i_-\mathsf{P}^i_+\right) \tag{8.6}$$

of the universal enveloping algebra $U(\mathfrak{n})$, expressed in terms of left or right isometry generators for the action of the isometry group $\mathcal{N}_\mathrm{L} \times \mathcal{N}_\mathrm{R}$ on NW_6 [24, 38, 45].

However, in the manner which we have constructed things, this is not the case. Recall that the approximation in which our quantization of the geometry of NW_6 holds is the small time limit $x^+ \to 0$ in which the plane wave approaches flat six-dimensional Minkowski space $\mathbb{E}^{1,5}$. To incorporate the effects of the curved geometry of NW_6 into our formalism, we have to replace the derivative operators $\widetilde{\partial}^a_\star$ appearing in (8.1) with appropriate curved space analogues $\delta^a_\star$ [6, 42].

Recall that the derivative operators $\partial^a_\star$ are *not* derivations of the star product $\star$, but instead obey the deformed Leibniz rules (5.12). The deformation arose from twisting the co-action of the bialgebra $U(\mathfrak{g})$ so that it generated automorphisms of the noncommutative algebra of functions, i.e. isometries of the noncommutative plane wave. The basic idea is to now àbsorb' these twistings into derivations $\delta^a_\star$ obeying the usual Leibniz rule

$$\delta^a_\star \triangleright (f \star g) = (\delta^a_\star \triangleright f) \star g + f \star (\delta^a_\star \triangleright g). \tag{8.7}$$

These derivations generically act on $C^\infty(\mathfrak{n}^\vee)$ as the noncommutative $\star$-polydifferential operators

$$\delta^a_\star \triangleright f = \sum_{n=1}^{\infty} \xi_a{}^{a_1\cdots a_n} \star \left(\partial^\star_{a_1} \triangleright \cdots \triangleright \partial^\star_{a_n} \triangleright f\right) \tag{8.8}$$

with $\xi_a{}^{a_1\cdots a_n} \in C^\infty(\mathfrak{n}^\vee)$. Unlike the derivatives $\partial^a_\star$, these derivations will no longer star commute among each other. There is a one-to-one correspondence [47] between such derivations $\delta^\star_a$ and Poisson vector fields $E^a = E^a{}_b\partial^b$ on $\mathfrak{n}^\vee$ obeying

$$E^a \circ \Theta(f, g) = \Theta(E^a f, g) + \Theta(f, E^a g) \tag{8.9}$$

for all $f, g \in C^\infty(\mathfrak{n}^\vee)$. To leading order one has $\delta^a_\star \triangleright f = E^a{}_b \star (\partial^b_\star \triangleright f) + O(\theta)$. By identifying the Lie algebra $\mathfrak{n}$ with the tangent space to NW_6, at this order the vector fields E^a can be thought of as defining a natural local frame with flat metric η_{ab} and a curved metric tensor $G^\star_{ab} = \frac{1}{2}\eta_{cd}\left(E^c{}_a \star E^d{}_b + E^d{}_a \star E^c{}_b\right)$ on the noncommutative space NW_6. However, for our star products there are always higher order terms in (8.8) which spoil this interpretation. The noncommutative frame fields $\delta^a_\star$ describe the *quantum* geometry of the plane wave NW_6. In particular, the metric tensor $G^\star$ will in general differ from the classical open string metric $G_{\rm open}$. While the operators $\delta^a_\star$ always exist as a consequence of the Kontsevich formality map [6, 47], computing them explicitly is a highly difficult problem. We will see some explicit examples below, as we now begin to tour through our three star products. Throughout, we shall take the natural choice of measure $\kappa = \sqrt{|\det G|} = \frac{1}{2}$, the constant Riemannian volume density of the NW_6 plane wave geometry.

8.1. Time ordering

In the case of time ordering, we use (6.6) to compute

$$\Box^* \triangleright \varphi = (2\partial_+\partial_- + \overline{\boldsymbol{\partial}} \cdot \boldsymbol{\partial})\varphi \tag{8.10}$$

and thus the equation of motion coincides with that of a free scalar particle on flat Minkowski space $\mathbb{E}^{1,5}$ (deviations from flat spacetime can only come about here by choosing a time-dependent measure κ_*). This illustrates the point made above that the treatment of the present paper tackles only the semi-classical flat space limit of the spacetime NW_6. The appropriate curved geometry for this ordering corresponds to the global coordinate system (2.5) in which the classical Laplace operator is given by

$$\Box^*_0 = 2\partial_+\partial_- + \left|\boldsymbol{\partial} + \frac{\mathrm{i}}{2}\theta\overline{\boldsymbol{z}}\partial_-\right|^2, \tag{8.11}$$

so that the free wave equation $(\Box^*_0 - m^2)\varphi = 0$ is equivalent to the Schrödinger equation for a particle of charge p^+ (the momentum along the x^- direction) in a constant magnetic field of strength θ. A global pseudo-orthonormal frame is provided by the commutative vector fields

$$\begin{aligned} E^*_- &= \partial_-, & E^*_+ &= \partial_+ - \mathrm{i}\theta(\boldsymbol{z}\cdot\boldsymbol{\partial} - \overline{\boldsymbol{z}}\cdot\overline{\boldsymbol{\partial}}), \\ E^i_* &= \partial^i, & \overline{E}^i_* &= \overline{\partial}^i. \end{aligned} \tag{8.12}$$

Determining the derivations δ^a_* corresponding to the commuting frame (8.12) on the quantum space is in general rather difficult. Evidently, from the coproduct structure (5.16) the action along the light-cone position is given by

$$\delta^*_- \triangleright f = \partial_- f. \tag{8.13}$$

This is simply a consequence of the fact that translations along x^- generate an automorphism of the noncommutative algebra of functions, i.e. an isometry of the noncommutative geometry. From the Hopf algebra coproduct (5.14) we have

$$\Delta_*(\mathrm{e}^{\mathrm{i}\theta\partial_-}) = \mathrm{e}^{\mathrm{i}\theta\partial_-} \otimes \mathrm{e}^{\mathrm{i}\theta\partial_-} \tag{8.14}$$

and consequently

$$\mathrm{e}^{\mathrm{i}\theta\partial_-} \triangleright (f * g) = (\mathrm{e}^{\mathrm{i}\theta\partial_-} \triangleright f) * (\mathrm{e}^{\mathrm{i}\theta\partial_-} \triangleright g). \tag{8.15}$$

On the other hand, the remaining isometries involve intricate twistings between the light-cone and transverse space directions. For example, let us demonstrate how to unravel the coproduct rule for ∂_+^* in (5.16) into the desired symmetric Leibniz rule (8.7) for δ_+^*. This can be achieved by exploiting the $*$-product identities

$$z_i * f = (\mathrm{e}^{\mathrm{i}\theta\partial_-} f) * z_i - 2\mathrm{i}\theta x^+ \overline{\partial}^i f, \qquad \overline{z}_i * f = (\mathrm{e}^{-\mathrm{i}\theta\partial_-} f) * \overline{z}_i + 2\mathrm{i}\theta x^+ \partial^i f \tag{8.16}$$

along with the commutativity properties $[\partial_-^*, z_i]_* = [\partial_-^*, \overline{z}_i]_* = 0$ for $i = 1, 2$ and for arbitrary functions f. Using in addition the modified Leibniz rules (5.16) along with the $*$-multiplication properties (6.5), we thereby find

$$\delta_+ \triangleright f = \left[x^+ \partial_+ + \frac{1}{2\mathrm{i}} (\boldsymbol{z} \cdot \overline{\boldsymbol{\partial}} + \overline{\boldsymbol{z}} \cdot \boldsymbol{\partial}) \right] f. \tag{8.17}$$

This action mimics the form of the classical frame field E_+^* in (8.12).

Finally, for the transversal isometries, one can attempt to seek functions $g^i \in C^\infty(\mathfrak{n}^\vee)$ such that $g^i * f = (\mathrm{e}^{-\mathrm{i}\theta\partial_-} f) * g_i$ in order to absorb the light-cone translation in the Leibniz rule for ∂_*^i in (5.16). This would mean that the x^- translations are generated by *inner* automorphisms of the noncommutative algebra. If such functions exist, then the corresponding derivations are given by $\delta_*^i \triangleright f = g^i * \partial_*^i f$ (no sum over i) and similarly for $\overline{\delta}_*^i$. However, it is doubtful that such inner derivations exist and the transverse space frame fields are more likely to be given by higher order $*$-polyvector fields. For example, using similar steps to those which led to (8.17), one can show that the actions

$$\begin{aligned} \delta_* \triangleright f &:= (\overline{\boldsymbol{z}} \cdot \boldsymbol{\partial} + 2\mathrm{i}x^+ \partial_+ - \mathrm{i}\theta x^+ \overline{\boldsymbol{\partial}} \cdot \boldsymbol{\partial}) f, \\ \overline{\delta}_* \triangleright f &:= (\boldsymbol{z} \cdot \overline{\boldsymbol{\partial}} - 2\mathrm{i}x^+ \partial_+ + \mathrm{i}\theta x^+ \overline{\boldsymbol{\partial}} \cdot \boldsymbol{\partial}) f \end{aligned} \tag{8.18}$$

define derivations of the $*$-product on NW_6, and hence naturally determine elements of a noncommutative transverse frame.

The action of the corresponding noncommutative Laplacian $\eta_{ab}\delta_*^a \triangleright (\delta_*^b \triangleright \varphi)$ deforms the harmonic oscillator dynamics generated by (8.11) by non-local higher spatial derivative terms. These extra terms will have significant ramifications at large energies for motion in the transverse space. This could have profound physical effects in the interacting noncommutative quantum field theory. In particular, it may alter the UV/IR mixing story [51] in an interesting way. For time-dependent noncommutativity with standard tree-level propagators, UV/IR mixing becomes intertwined with violations of energy conservation in an intriguing way [7, 56], and it would be interesting to see how our modified free field propagators affect this analysis. It would also be interesting to see if and how these modifications are related to the generic connection between wave propagation on homogeneous plane waves and the Lewis–Riesenfeld theory of time-dependent harmonic oscillators [10].

8.2. Symmetric time ordering

The analysis in the case of symmetric time ordering is very similar to that just performed, so we will be very brief and only highlight the essential changes. From (6.9) we find once again that the Laplacian (8.5) coincides with the flat space wave operator

$$\square^\bullet \triangleright \varphi = (2\partial_+\partial_- + \overline{\boldsymbol{\partial}} \cdot \boldsymbol{\partial})\varphi. \tag{8.19}$$

The relevant coordinate system in this case is given by the Brinkman metric (2.6) for which the classical Laplace operator reads

$$\square_0^\bullet = 2\partial_+\partial_- + \overline{\boldsymbol{\partial}} \cdot \boldsymbol{\partial} - \tfrac{1}{4}\theta^2 |\boldsymbol{z}|^2 \partial_-^2. \tag{8.20}$$

A global pseudo-orthonormal frame in this case is provided by the vector fields

$$E_-^\bullet = \partial_-, \qquad E_+^\bullet = \partial_+ + \tfrac{1}{8}\theta^2|\boldsymbol{z}|^2\partial_-, \qquad E_\bullet^i = \partial^i, \qquad \overline{E}_\bullet^i = \overline{\partial}^i. \tag{8.21}$$

The corresponding twisted derivations $\delta_\bullet^a$ which symmetrize the Leibniz rules (5.19) can be constructed analogously to those of the time ordering case in section 8.1.

8.3. Weyl ordering

Finally, the case of Weyl ordering is particularly interesting because the effects of curvature are present even in the flat space limit. Using (6.12) we find the Laplacian

$$\Box^\star \triangleright \varphi = \left(2\partial_+\partial_- + 2\left[2\left(1 - \frac{\sin(\theta\partial_-)}{\theta\partial_-}\right) + \frac{1-\cos(\theta\partial_-)}{\theta^2\partial_-^2}\right]\overline{\boldsymbol{\partial}}\cdot\boldsymbol{\partial}\right)\varphi \tag{8.22}$$

which coincides with the flat space Laplacian only at $\theta = 0$. To second order in the deformation parameter θ, the equation of motion (8.4) thereby yields a second-order correction to the usual flat space Klein–Gordan equation given by

$$\left[\left(2\partial_+\partial_- + \overline{\boldsymbol{\partial}}\cdot\boldsymbol{\partial} - m^2\right) + \tfrac{7}{12}\theta^2\partial_-^2\overline{\boldsymbol{\partial}}\cdot\boldsymbol{\partial} + O(\theta^4)\right]\varphi = 0. \tag{8.23}$$

Again we find that only the transverse space motion is altered by noncommutativity, but this time through a non-local dependence on the light-cone momentum p^+ yielding a drastic modification of the dispersion relation for free wave propagation in the noncommutative spacetime. This dependence is natural. The classical mass-shell condition for motion in the curved background is $2p^+p^- + |4\theta p^+\boldsymbol{\lambda}|^2 = m^2$, where $\boldsymbol{\lambda} \in \mathbb{C}^2$ represents the position and radius of the circular trajectories in the background magnetic field [24]. Thus, the quantity $4\theta p^+\boldsymbol{\lambda}$ can be interpreted as the momentum for motion in the transverse space. The operator (8.22) incorporates the appropriate noncommutative deformation of this motion. It illustrates the point that the fundamental quanta governing the interactions in the present class of noncommutative quantum field theories are not likely to involve the particle-like dipoles of the flat space cases [9, 59], but more likely string-like objects owing to the non-vanishing H-flux in (2.7). These open string quanta become polarized as dipoles experiencing a net force due to their couplings to the non-uniform B-field. It is tempting to speculate that, in contrast to the other orderings, the Weyl ordering naturally incorporates the new vacua corresponding to long string configurations which are due entirely to the time-dependent nature of the background Neveu–Schwarz field [8].

While the Weyl-ordered star product is natural from an algebraic point of view, it does not correspond to a natural coordinate system for the plane wave NW_6 due to the complicated form of the group product rule (3.45) in this case. In particular, the frame fields in this instance will be quite complicated. Computing the corresponding twisted derivations $\delta_\star^a$ directly would again be extremely cumbersome, but luckily we can exploit the equivalence between the star products $\star$ and $*$ derived in section 3.3. Given the derivations δ_*^a constructed in section 8.1, we may use the differential operator (3.44) which implements the equivalence (3.32) to define

$$\delta_\star^a \triangleright f := \mathcal{G}_\Omega \circ \delta_*^a \triangleright \left(\mathcal{G}_\Omega^{-1}(f)\right). \tag{8.24}$$

These noncommutative frame fields will lead to the appropriate curved space extension of the Laplace operator in (8.22).

9. Worldvolume field theories

In this final section, we will describe how to build noncommutative field theories on regularly embedded worldvolumes of D-branes in the spacetime NW_6 using the formalism described above. We shall describe the general technique on a representative example by comparing the noncommutative field theory on NW_6 which we have constructed in this paper to that of the noncommutative D3-branes which was constructed in [38]. We shall do so in a general fashion which illustrates how the construction extends to generic D-branes. This will provide further perspective on the natures of the different quantizations we have used throughout, and also illustrate the overall consistency of our results. As we will now demonstrate, we can view the noncommutative geometry of NW_6, in the manner constructed above, as a collection of all Euclidean noncommutative D3-branes taken together. This is done by restricting the geometry to obtain the usual quantization of coadjoint orbits in $\mathfrak{n}^\vee$ (as opposed to all of $\mathfrak{n}^\vee$ as described above). This restriction defines an alternative and more geometrical approach to the quantization of these branes which does not rely upon working with representations of the Lie group $\mathcal{N}$, and which is more adapted to the flat space limit $\theta \to 0$. This procedure can be thought of as somewhat opposite to the philosophy of [38], which quantized the geometry of a non-symmetric D5-brane wrapping NW_6 [48] by viewing it as a noncommutative foliation by these Euclidean D3-branes. Here the quantization of the spacetime-filling brane in NW_6 has been carried out independently leading to a much simpler noncommutative geometry which correctly induces the anticipated worldvolume field theories on the $\mathbb{E}^4$ submanifolds of NW_6.

The Euclidean D3-branes of interest wrap the non-degenerate conjugacy classes of the group $\mathcal{N}$ and are coordinatized by the transverse space $z \in \mathbb{C}^2 \cong \mathbb{E}^4$ [61]. They are defined by the spacelike hyperplanes of constant time in NW_6 given by the transversal intersections of the null hypersurfaces

$$x^+ = \text{constant}, \qquad x^- + \tfrac{1}{4}\theta|z|^2 \cot\left(\tfrac{1}{2}\theta x^+\right) = \text{constant}, \tag{9.1}$$

independently of the chosen coordinate frame. This describes the brane worldvolume as a wavefront expanding in a sphere S^3 in the transverse space. In the semi-classical flat space limit $\theta \to 0$, the second constraint in (9.1) to leading order becomes

$$C := 2x^+x^- + |z|^2 = \text{constant}. \tag{9.2}$$

The function C on $\mathfrak{n}^\vee$ corresponds to the Casimir element (8.6) and the constraint (9.2) is analogous to the requirement that Casimir operators act as scalars in irreducible representations. Similarly, the constraint on the time coordinate x^+ in (9.1) is analogous to the requirement that the central element T acts as a scalar operator in any irreducible representation of $\mathcal{N}$.

Let $\pi : NW_6 \to \mathbb{E}^4$ be the projection of the six-dimensional plane wave onto the worldvolume of the symmetric D3-branes. Let $\pi^\sharp : C^\infty(\mathbb{E}^4) \to C^\infty(NW_6)$ be the induced algebra morphism defined by pull-back $\pi^\sharp(f) = f \circ \pi$. To consistently reduce the noncommutative geometry from all of NW_6 to its conjugacy classes, we need to ensure that the candidate star product on $\mathfrak{n}^\vee$ respects the Casimir property of the functions x^+ and C, i.e. that x^+ and C star commute with every function $f \in C^\infty(\mathfrak{n}^\vee)$. Only in that case can the star product be consistently restricted from all of NW_6 to a star product $\star_{x^+}$ on the conjugacy classes $\mathbb{E}^4$ defined by

$$f \star_{x^+} g := \pi^\sharp(f) \star \pi^\sharp(g). \tag{9.3}$$

Then one has the compatibility condition

$$\iota^\sharp(f \star g) = \iota^\sharp(f) \star_{x^+} \iota^\sharp(g), \tag{9.4}$$

where $\iota^\sharp : C^\infty(NW_6) \to C^\infty(\mathbb{E}^4)$ is the pull-back induced by the inclusion map $\iota : \mathbb{E}^4 \hookrightarrow NW_6$. In this case, one has an isomorphism $C^\infty(\mathbb{E}^4) \cong C^\infty(NW_6)/\mathcal{J}$ of associative noncommutative algebras [12], where $\mathcal{J}$ is the two-sided ideal of $C^\infty(NW_6)$ generated by the Casimir constraints ($x^+ -$ constant) and ($C -$ constant). This procedure is a noncommutative version of Poisson reduction, with the Poisson ideal $\mathcal{J}$ implementing the geometric requirement that the Seiberg–Witten bi-vector Θ be tangent to the conjugacy classes.

From the star commutators (7.11), (7.15) and (7.20) we see that $[x^+, f]_\star = 0$ for all three of our star products. However, the condition $[C, f]_\star = 0$ is *not* satisfied. Although classically one has the Poisson commutation $\Theta(C, f) = 0$, one can only consistently restrict the star products by first defining an appropriate projection of the algebra of functions on $\mathfrak{n}^\vee$ onto the star subalgebra $\mathcal{C}$ of functions which star commute with the Casimir function C. One easily computes that $\mathcal{C}$ naturally consists of functions f which are independent of the light-cone position, i.e. $\partial_- f = 0$. Then the projection $\iota^\sharp$ above may be applied to the subalgebra $\mathcal{C}$ on which it obeys the requisite compatibility condition (9.4). The general conditions for reduction of Kontsevich star products to D-submanifolds of Poisson manifolds are described in [18, 20].

With these projections implicitly understood, one straightforwardly finds that all three star products (3.23), (3.28) and (3.47) restrict to

$$f \star_{x^+} g = \mu \circ \exp[\mathrm{i}\theta x^+(\boldsymbol{\partial}^\top \otimes \overline{\boldsymbol{\partial}} - \overline{\boldsymbol{\partial}}^\top \otimes \boldsymbol{\partial})] f \otimes g \tag{9.5}$$

for functions $f, g \in C^\infty(\mathbb{E}^4)$. This is just the Moyal product, with noncommutativity parameter θx^+, on the noncommutative Euclidean D3-branes. It is cohomologically equivalent to the Voros product which arises from quantizing the conjugacy classes through endomorphism algebras of irreducible representations of the twisted Heisenberg algebra $\mathfrak{n}$, with a normal or Wick ordering prescription for the generators $\mathsf{P}^i_\pm$ [38]. In this case, the noncommutative Euclidean space arises from a projection of $U(\mathfrak{n})$ in the discrete representation V^{p^+,p^-} whose second Casimir invariant (8.6) is given in terms of light-cone momenta as $\mathsf{C} = -2p^+(p^- + \theta)$ and with $\mathsf{T} = \theta p^+$. In this approach, the noncommutativity parameter is naturally the *inverse* of the effective magnetic field $p^+\theta$. On the other hand, the present analysis is a more geometrical approach to the quantization of symmetric D3-branes in NW_6 which deforms the Euclidean worldvolume geometry by the time parameter θx^+ without resorting to endomorphism algebras. The relationship between the two sets of parameters is given by $x^+ = p^+\tau$, where τ is the proper time coordinate for geodesic motion in the pp-wave geometry of NW_6.

In contrast to the coadjoint orbit quantization [38], the noncommutativity found here matches exactly that predicted from string theory in the semi-classical limit [28], which asserts that the Seiberg–Witten bi-vector on the D3-branes is given by $\Theta_{x^+} = \frac{\mathrm{i}}{2}\sin(\theta x^+)\boldsymbol{\partial}^\top \wedge \overline{\boldsymbol{\partial}}$. Note that the present analysis also covers as a special case the degenerate cylindrical null branes located at time $x^+ = 0$ [61], for which (9.5) becomes the ordinary pointwise product $f \star_0 g = fg$ of worldvolume fields and as expected these branes support a *commutative* worldvolume geometry. In contrast, the commutative null branes correspond to the class of continuous representations of the twisted Heisenberg algebra having quantum number $p^+ = 0$ which must be dealt with separately [38].

It is elementary to check that the rest of the geometrical constructs of this paper reduce to the standard ones appropriate for a Moyal space. By defining

$$\partial^a_{\star_{x^+}} \triangleright f := \iota^\sharp \circ \partial^a_\star \triangleright (\pi^\sharp(f)), \tag{9.6}$$

one finds that the actions of the derivatives constructed in section 6 all reduce to the standard ones of flat noncommutative Euclidean space, i.e. $\partial^i_{\star_{x^+}} \triangleright f = \partial^i f$, $\overline{\partial}^i_{\star_{x^+}} \triangleright f = \overline{\partial}^i f$ for $f \in C^\infty(\mathbb{E}^4)$. From section 5, one recovers the standard Hopf algebra of these derivatives with trivial coproducts $\Delta_{\star_{x^+}}$ defined by

$$\Delta_{\star_{x^+}}(\nabla_{\star_{x^+}}) \triangleright (f \otimes g) := (\iota^\sharp \otimes \iota^\sharp) \circ \Delta_\star(\nabla_\star) \triangleright (\pi^\sharp(f) \otimes \pi^\sharp(g)), \tag{9.7}$$

and hence the symmetric Leibniz rules appropriate to the translational symmetry of field theory on Moyal space. Consistent with the restriction to the conjugacy classes, one also has $\partial_{\pm}^{\star_{x^+}} \triangleright f = 0$.

However, from (5.15), (5.18) and (5.21) one finds a non-vanishing co-action of time translations given by

$$\Delta_{\star_{x^+}}\big(\partial_+^{\star_{x^+}}\big) = \theta\big(\boldsymbol{\partial}_{\star_{x^+}}{}^{\top} \otimes \overline{\boldsymbol{\partial}}_{\star_{x^+}} - \overline{\boldsymbol{\partial}}_{\star_{x^+}}{}^{\top} \otimes \boldsymbol{\partial}_{\star_{x^+}}\big). \tag{9.8}$$

This formula is very natural. The isometries of NW_6 in $\mathfrak{g} = \mathfrak{n}_L \oplus \mathfrak{n}_R$ corresponding to the number operator J of the twisted Heisenberg algebra are generated by the vector fields [38] $J_L = \theta^{-1}\partial_+$ and $J_R = -\theta^{-1}\partial_+ - \mathrm{i}(\boldsymbol{z} \cdot \boldsymbol{\partial} - \overline{\boldsymbol{z}} \cdot \overline{\boldsymbol{\partial}}) = \theta^{-1}E_+^*$ (in Brinkman coordinates). The vector field $J_L + J_R$ generates rigid rotations in the transverse space. Restricted to the D3-brane worldvolume, the time translation isometries thus truncate to rotations of $\mathbb{E}^4$ in $so(4)$. The coproduct (9.8) gives the standard twisted co-action of rotations for the Moyal algebra which define quantum rotational symmetries of noncommutative Euclidean space [21, 22, 64]. This discussion also drives home the point made earlier that our derivative operators $\partial_\star^a$ indeed do generate, through their twisted co-actions (Leibniz rules), quantum isometries of the full noncommutative plane wave.

Finally, a trace on $C^\infty(\mathbb{E}^4)$ is induced from (7.3) by restricting the integral to the submanifold $\iota : \mathbb{E}^4 \hookrightarrow NW_6$ and using the induced measure $\iota^\sharp(\kappa)$. For the measures constructed in section 7, $\iota^\sharp(\kappa)$ is always a constant function on $\mathbb{E}^4$ and hence the integration measures all restrict to the constant volume form of $\mathbb{E}^4$. Thus, noncommutative field theories on the spacetime NW_6 consistently truncate to the anticipated worldvolume field theories on noncommutative Euclidean D3-branes in NW_6, together with the correct twisted implementation for the action of classical worldvolume symmetries. The advantage of the present point of view is that many of the novel features of these canonical Moyal space field theories naturally originate from the pp-wave noncommutative geometry when the Moyal space is regarded as a regularly embedded coadjoint orbit in $\mathfrak{n}^\vee$, as described above. Furthermore, the method detailed in this paper allows a more systematic construction of the deformed worldvolume field theories of *generic* D-branes in NW_6 in the semi-classical regime, and not just the symmetric branes analysed here. For instance, the analysis can in principle be applied to describe the dynamics of symmetry-breaking D-branes which localize along products of twisted conjugacy classes in the Lie group $\mathcal{N}$ [54]. However, these branes have yet to be classified in the case of the gravitational wave NW_6.

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